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THE IMPACT OF A RIGID CIRCULAR CYLINDER ON AN ELASTIC SOLID

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CONTENTS

A method is described for approximating to any degree of accuracy the solution of the following problem: An elastic body which is bounded by a plane on one side, but extends to infinity otherwise, is hit by a circular disk of given mass, radius, and initial speed perpendicular to the plane boundary. The whole surface of the disk enters into contact with the elastic body at the same time and stays in contact at all its points from then on. The disk is assumed to be rigid, i.e. it does not allow the particles of the elastic body in the contact area to move relative to each other in a direction perpendicular to the plane boundary. For the relative motion of these particles parallel to the face of the disk several conditions are considered, representing perfect lubrication, various degrees of viscous friction and perfect adherence.

With the help of various Mellin transformations a method is indicated which leads to an expansion of the motion in powers of the Laplace transform variable. The case of perfect adherence needs some special consideration, and a simple approximation for the static problem is found. The case of perfect lubrication is then treated in more detail by a different method which replaces the condition of constant normal displacement in the contact area by an equivalent number of requirements for certain averages over the normal displacement in the contact area. The condition of rigidity for the disk is not exactly satisfied, but it is possible to judge the accuracy of the approximation (with the help of asymptotic expansions in the Laplace transform variable) at the initial time, when discrepancies are largest.

The concept of 'mode of vibration' is introduced. Any transient in the coupled system of elastic body and rigid disk can be described as superposition of modes, each of which is an exponentially damped harmonic oscillation of the coupled system with a frequency and damping constant independent of the particular transient. The motion of the impinging disk is then seen to be dominated mostly by the lowest mode, provided the mass of the disk is not too small. The displacement perpendicular to the boundary outside of the contact area has been calculated. This calculation is not more difficult than the corresponding one in the case of a point-like source at the plane boundary of an elastic solid.

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Numerical computations were carried out for the case of perfect lubrication with the help of the Elecom digital computer in order to determine the first two modes and their contributions to the motion of the disk. As long as Poisson's ratio for the elastic solid exceeds 1/4, the results do not depend strongly on the value of Poisson's ratio. The ratio of the areal mass densities of the disk to the elastic solid is the main parameter of the theory. The shear wave velocity of the elastic solid determines the time scale of the motion.

Introduction

The present work is concerned with the following problem of the theory of elasticity.

Let us consider an elastic body which is bounded by a plane on one side and stretches out to infinity otherwise. Its boundary is hit by a circular disc at the time $t = 0$ in such a way that the whole surface of the disk enters into contact with the boundary of the elastic body at the same time. The circular disk of radius r_0 is assumed to have a known mass M and is supposed to be rigid, i.e. it forces the particles on the boundary of the elastic body in the area of contact at each moment to have the same displacement perpendicular to the boundary. The condition of rigidity for the disk affects only the area of contact and the displacement of the particles there in the direction perpendicular to the boundary. It is also assumed in the mathematical treatment of this problem that this condition of rigidity for the circular disk persists during all later times $t > 0$. It can be said, therefore, that the circular disk adheres to the particles on the boundary of the elastic body in such a way that these particles can have no motion relative to each other in a direction perpendicular to the plane boundary. Concerning the relative motion of the particles in the area of contact in a direction parallel to the plane boundary, a number of conditions will be considered. These conditions represent not only the two extreme cases of perfect lubrication and of perfect adherence but also the whole range of intermediate cases of viscous friction between the rigid circular disk and the particles of the elastic solid in the area of contact. The circular disk is assumed to have a speed w perpendicular to the plane boundary as it hits the elastic solid. As the plane boundary of the elastic solid is assumed to be horizontal, the gravitational force Mg acting on the circular disk is taken into account in addition to the total normal stress in the contact area.

The case of perfect lubrication has been treated previously to some extent. But instead of the impact problem as presented above, the corresponding steady-state problem was investigated. The circular disk was assumed to be in contact with the plane boundary of the elastic solid at all times, not only starting at $t = 0$, and a periodic external driving force was considered. Wolf (1944) assumed that the distribution of normal stress in the contact area was the same as in the static limit, i.e. when the frequency of the driving force became zero. He was then able to compute the distribution of the normal displacement in the contact area for a given frequency of the external driving force. The average of this distribution was used as an approximation to the normal displacement of the rigid disk with the correct boundary condition (normal displacement constant in the contact area). Bycroft (1956) uses the same approximation, but he shows, in addition, that Wolf's assumption concerning the stress distribution in the contact area yields a useful upper and lower bound for the normal displacement with the correct boundary conditions. Indeed, if the stress distribution in the contact area is assumed to be the same as in the corresponding static problem, then the resulting normal displacement at the rim of the disk is too small, whereas the weighted average over the resulting distribution of normal displacement with a weighting function $r(r_0^2-r^2)^{-\frac{1}{2}}$ is too large. Miller & Pursey (1954) assumed constant normal stress in the contact area and computed the average normal displacement. The assumption of constant normal stress in the contact area is, in a way, the opposite of Wolf's assumption of the static stress distribution. Indeed, a constant normal stress results at the first moment if the rigid disk is given a very short mechanical impulse. It is only natural that the results of these other investigations can also be obtained with the present methods. However, no detailed discussion of these previous results will be made.

The solution of the problem stated above can be approximated in various ways. For instance, the method of either Bycroft or Miller & Pursey yields approximations to the non-stationary problem which is considered presently. The purpose of this work is to find a scheme of approximations which has the following qualities. (1) It can be carried to any desired degree of accuracy; (2) it gives the excitation of the elastic body outside the area of contact with the impinging disk; and (3) its formulae can easily be programmed for a high-speed computer. In accordance with these goals it was found necessary to discuss in some detail the mathematics involved in the scheme of approximation. Indeed, the mathematical apparatus involves a good deal of the theory of functions of a complex variable which cannot easily be replaced by physical intuition. An effort is made to find a precise statement for the degree of accuracy with which the conditions of the problem are satisfied. Numerical calculations have been performed in order to have explicit results for some of the simpler quantities which can be visualized immediately. But the various parameters of the theory are not taken through their full range, nor is there a sample computation for every quantity which is expressed by a formula. On the other hand, there are a few conjectural statements which might have been substantiated by extensive computations or by refined mathematics. Unfortunately, the author did not have the skill for either, and he asks for forgiveness since none of these statements serves as a hypothesis for any following derivation. The content of this paper will be described in the following paragraphs in order to help the reader in finding his way through the many mathematical details.

The basic equations of the problem are given in §1. A Laplace transformation is applied to the time variable t, and a Hankel transformation to the radial variable r. The boundary conditions are chosen so as to describe various situations in the area of contact. The main special cases are perfect lubrication (vanishing shear stress in the area of contact) and perfect adherence (vanishing tangential displacement in the area of contact) between which there are various degrees of viscous friction (shear stress proportional to tangential particle velocity in the area of contact.)

By a number of Mellin transformations the problem is formulated in §2 as a relation between analytic functions whose singularities in certain parts of the complex plane are qualitatively known. In the case of perfect lubrication this leads to a recursive scheme for finding an expansion of the solution in ascending powers of the Laplace transform variable p (descending powers of the time variable t), but only certain qualitative features of this expansion are retained for the further development.

The solution for the case of perfect adherence can also be expanded in ascending powers of p, as will be shown in §3, but starting with the quadratic term, the logarithm of p will also appear. Moreover, an infinite set of linear equations has to be solved for each power of p. This set is solved by a convergent expansion in powers of $(b/a)^4$, where b is the shear wave velocity and a the velocity of the compressional waves. This method is applied to the stationary problem ($p = 0$) of perfect adherence, and gives a good approximation in terms of elementary functions.

The problem of the impinging disk with perfect lubrication is formulated differently in § 4. A finite expansion of $N+1$ terms is assumed for the normal stress σ in the contact area, with each term ($n = 0, 1, \ldots, N$) characterized by its dependence $(1-r^2/r_0^2)^{n-\frac{1}{2}}$ on the radial variable r (with r_0 the radius of the impinging disk). Such an expansion was previously (§ 2) shown to be equivalent to an expansion in powers of p up to p^{2N}. The normal displacement u can then be expressed directly in terms of the assumed expansion of σ. As a check, the proportionality between the normal particle velocity and the normal stress in the area of contact at the time of impact $t = 0$ is established.

The coefficients in the expansion of σ are determined in § 5 by imposing $N+1$ requirements on the corresponding displacement u. The zero-order requirement insures the balance between the inertia and weight of the impinging disk on one hand, and the sum of the stresses σ in the contact area on the other. The remaining N requirements tend to make the normal displacement u in the contact area as flat as possible, by making u orthogonal to a certain set of N polynomials. The quality of this procedure can be expected to be worst at $t = 0$. Therefore, the normal particle velocity as given by this method at time $t = 0$ is established with the help of certain asymptotic expansions, for $N = 0, 1, 2, 3, 4$. This constitutes the main criterion for judging the accuracy of this scheme of approximation as a function of N.

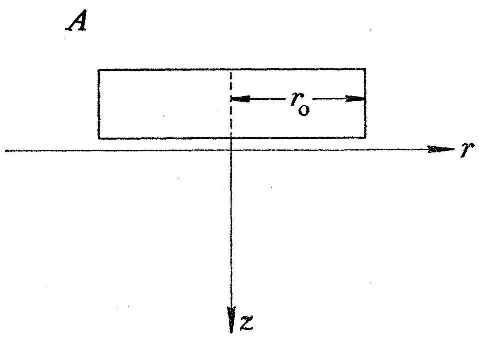
The motion of the disk as a function of t is obtained in § 6 by inverting the Laplace transformation with the help of the standard procedure of integrating along the imaginary axis of the complex p plane. This integral can be reduced to a sum over the residues of poles in the negative p plane. The locations of these poles, i.e. the complex frequencies associated with them, depend only on the boundary conditions, but not on the particular manner in which the disk is set into motion, e.g. by impinging at a given speed w. The motion of the disk appears therefore as a superposition of exponentially damped, harmonic oscillations which are called 'modes of vibration.' Each mode is associated with an exponentially damped, harmonic oscillation of the whole elastic body which satisfies all the boundary conditions.

The disturbance which travels along the boundary of the elastic body outside the area of contact is considered in § 7. The method of computation is essentially that of Cagniard (1939), but the source of the disturbance is now extended over the whole contact area, whereas usually only point-like sources are considered. The derivation is somewhat complicated for the terms $n = 1, \ldots, N$ in the expression of σ, although the result is not any more involved than for the term $n = 0$.

Finally, § 8 is devoted to the numerical work on the problem. All the transcendental functions which occur in this problem are entire functions of p. It is therefore natural to use their power series expansions, in particular since their coefficients are obtained by three-term recursion formulae. The first two modes are obtained from an expansion of σ with only three terms ($N = 2$) for a set of parameters a^2/b^2 and a certain ratio q of areal mass densities involved which is the only other parameter of the problem as long as the potential energy of the disk due to its weight is neglected compared to the kinetic energy at impact. Various results are plotted, and the period and damping constant of the lowest mode are given in a short table.

1. The mixed boundary-value problem

A cylindrical co-ordinate system is used, such that the semi-infinite elastic body is at $z > 0$ and the centre of the impinging disk at $r = 0$. Because of the cylindrical symmetry of the problem, there is no angular dependence, and it is assumed that all displacements are restricted to the rz planes. This leaves only the axial (vertical) displacement u and the radial (horizontal) displacement v to be considered.



Lamé's constants for the elastic medium are called λ and μ; a is the velocity of the compressional waves, b is the velocity of the distortional waves. In terms of Poisson's ratio, ν,

$$k^2 = \frac{a^2}{b^2} = \frac{\lambda+2\mu}{\mu} = 2\frac{1-\nu}{1-2\nu}. \tag{1}$$

Two potentials ϕ and ψ are introduced by

$$u = \frac{\partial\phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\psi), \quad v = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}. \tag{2}$$

With the symbols

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2}, \quad \omega = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}r\psi\right) + \frac{\partial^2\psi}{\partial z^2}, \tag{3}$$

the equations of motion for the elastic body are

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial t^2} - a^2\Delta\right) - \frac{\partial}{\partial z}\left(\frac{\partial^2\psi}{\partial t^2} - b^2\omega\right) &= 0, \\ \frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2} - a^2\Delta\right) + \frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial^2\psi}{\partial t^2} - b^2\omega\right) &= 0. \end{aligned}\right\} \tag{4}$$

The normal stress σ and the shear stress τ at the boundary are

$$\left.\begin{aligned} \sigma &= (\lambda+2\mu)\,\Delta - 2\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \frac{1}{r}\frac{\partial^2}{\partial r\,\partial z}(r\psi)\right], \\ \tau &= \mu\omega + 2\mu\left(\frac{\partial^2\phi}{\partial z\,\partial r} - \frac{\partial^2\psi}{\partial z^2}\right). \end{aligned}\right\} \tag{5}$$

It is assumed that the impinging disk gets into contact with the semi-infinite elastic solid at time $t = 0$, and that there is no strain in the elastic solid prior to this time. If it is assumed in addition, that for each time $t > 0$ the disturbance has not yet advanced beyond a certain finite depth z, then the equation (4) reduces to

$$\frac{\partial^2\phi}{\partial t^2} - a^2\Delta = 0, \quad \frac{\partial^2\psi}{\partial t^2} - b^2\omega = 0. \tag{6}$$

As all events start at time $t = 0$, it is natural to consider the one-sided p-multiplied Laplace transforms (indicated by attaching a star to the symbol) of the quantities involved, e.g.

$$\phi^* = p\int_0^\infty \phi e^{-pt} dt.$$

A useful solution of (6) is then

$$\phi^* = \int_0^\infty \Phi(\rho, p)\, J_0\left(\frac{pr}{b}\rho\right) e^{-(pz/b)\alpha} d\rho, \quad \alpha = \sqrt{(k^{-2}+\rho^2)};$$

$$\psi^* = \int_0^\infty \Psi(\rho, p)\, J_1\left(\frac{pr}{b}\rho\right) e^{-(pz/b)\beta} d\rho, \quad \beta = \sqrt{(1+\rho^2)}. \tag{7}$$

J_0 and J_1 are the Bessel functions of order 0 and 1. From (7) and (2) it follows that for $z = 0$

$$\left.\begin{aligned} u^* &= \frac{p}{b}\int_0^\infty (-\alpha\Phi + \rho\Psi)\, J_0\left(\frac{pr}{b}\rho\right) d\rho, \\ v^* &= \frac{p}{b}\int_0^\infty (-\rho\Phi + \beta\Psi)\, J_1\left(\frac{pr}{b}\rho\right) d\rho. \end{aligned}\right\} \tag{8}$$

Similar formulas follows from (7) and (5) for σ^* and τ^*. If Hankel's theorem (see Watson 1952) is applied, the functions Φ and Ψ can be expressed in terms of σ^* and τ^* by

$$\left.\begin{aligned} \mu[(1+2\rho^2)^2 - 4\rho^2\alpha\beta]\,\Phi &= \rho\,(1+2\rho^2)\int_0^\infty \sigma^*\, J_0\left(\frac{pr}{b}\rho\right) r\,dr - 2\rho^2\beta\int_0^\infty \tau^* J_1\left(\frac{pr}{b}\rho\right) r\,dr, \\ \mu[(1+2\rho^2)^2 - 4\rho^2\alpha\beta]\,\Psi &= 2\rho^2\alpha\int_0^\infty \sigma^* J_0\left(\frac{pr}{b}\rho\right) r\,dr - \rho(1+2\rho^2)\int_0^\infty \tau^* J_1\left(\frac{pr}{b}\rho\right) r\,dr, \end{aligned}\right\} \tag{9}$$

where σ^* and τ^* are evaluated at $z = 0$. By means of (8) and (9) the displacements at the boundary $z = 0$ of the elastic solid can be directly computed from the stresses at the boundary. Equations (8) and (9) are the basis for all further arguments.

The problem of the rigid circular disk of radius r_0 and mass M impinging with the speed w on the plane boundary of a semi-infinite elastic solid is now stated in the following manner. It is required to find stress distributions σ^* and τ^* in such a way that

$$\sigma^* = 0, \quad \tau^* = 0 \quad \text{for} \quad r > r_0, \tag{10}$$

$$u^* = h(p) \text{ independent of } r \quad \text{for} \quad r < r_0, \tag{11}$$

$$D\mu p v^* = (1-D)\, b\tau^* \quad \text{for} \quad r < r_0, \tag{12}$$

$$M[p^2h(p) - pw] = 2\pi\int_0^{r_0} \sigma^*\, r\,dr + Mg; \tag{13}$$

where u^* and v^* are given by means of (8) and (9) in terms of σ^* and τ^*. $h(p)$ describes the motion of the disk. This motion is determined by condition (13), if it is assumed that for all times $t > 0$ the disk stays in contact with the boundary of the elastic body at all its points. Condition (12) describes the radial motion of the boundary of the elastic body in the contact area with the help of a friction coefficient D whose values are restricted by the inequality $0 \leqslant D \leqslant 1$.

2. Application of the Mellin transformation

The theory of the Mellin transformation is used very frequently in this section. For convenience some formulae are listed below (for a proof see Titchmarsh 1937).

The Mellin transform $g(s)$ of $f(x)$ is given by

$$g(s) = \int_0^\infty f(x)\, x^{s-1}\,\mathrm{d}x, \tag{14}$$

which is supposed to converge for all complex s in a strip $c' < \mathscr{R}\, s = c < c''$. The inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} g(s)\, x^{-s}\,\mathrm{d}s \tag{15}$$

holds for all real c such that $c' < c < c''$. An integration such as (15) will usually be written without limits on the integral sign, because the factor in front of the integral sign implies that the path of integration in the complex s plane is parallel to the imaginary axis with a real part $\mathscr{R}\, s = c$ subject to some lower and upper bounds c' and c''. For example,

$$\int_0^\infty f_1(x) f_2(x)\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}}\int g_1(1-s)\, g_2(s)\,\mathrm{d}s \tag{16}$$

for $\mathscr{R}\, s = c$, provided the Mellin integral (14) converges with $f_1(x)$ for $\mathscr{R}\ s = 1-c$ and with $f_2(x)$ for $\mathscr{R}\ s = c$, the results being $g_1(s)$ and $g_2(s)$. Of special importance in this investigation is the Mellin transform of the Bessel function $J_n(\xi x)$ which is given for $-n < \mathscr{R}\, s < \frac{3}{2}$ by the formula

$$\int_0^\infty J_n(\xi x)\, x^{s-1}\,\mathrm{d}x = \frac{2^{s-1}\Gamma[\frac{1}{2}(s+n)]}{\xi^s\,\Gamma[\frac{1}{2}(n-s)+1]}, \tag{17}$$

where the gamma function occurs on the right. The relation between u, v and σ, τ is simpler when Mellin transforms are considered. Therefore define $\bar{\phi}$ and $\bar{\psi}$ by

$$\bar{\phi} = \int_0^\infty (\alpha\Phi - \rho\Psi)\,\rho^{-s}\,\mathrm{d}\rho, \quad \bar{\psi} = \int_0^\infty (\rho\Phi - \beta\Psi)\,\rho^{-s}\,\mathrm{d}\rho \tag{18}$$

and apply (16) and (17) to (8). The result is

$$\begin{aligned} u^* &= -\frac{p}{b}\frac{1}{2\pi\mathrm{i}}\int \bar{\phi}\,\frac{2^{s-1}\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\left(\frac{pr}{b}\right)^{-s}\mathrm{d}s, \\ v^* &= -\frac{p}{b}\frac{1}{2\pi\mathrm{i}}\int \bar{\psi}\,\frac{2^{s-1}\Gamma[\frac{1}{2}(1+s)]}{\Gamma[\frac{1}{2}(3-s)]}\left(\frac{pr}{b}\right)^{-s}\mathrm{d}s. \end{aligned} \tag{19}$$

Therefore it follows from (14) and (15) that

$$\begin{aligned} \bar{u}(s) &= \int_0^\infty u^*\left(\frac{r}{r_0}\right)^{s-1}\frac{\mathrm{d}r}{r_0^2} = -r_0^{-2}\left(\frac{pr_0}{2b}\right)^{1-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\,\bar{\phi}, \\ \bar{v}(s) &= \int_0^\infty v^*\left(\frac{r}{r_0}\right)^{s-1}\frac{dr}{r_0^2} = -r_0^{-2}\left(\frac{pr_0}{2b}\right)^{1-s}\frac{\Gamma[\frac{1}{2}(1+s)]}{\Gamma[\frac{1}{2}(3-s)]}\,\bar{\psi}. \end{aligned} \tag{20}$$

Again, define the Mellin transforms

$$\bar{\sigma}(s) = \int_0^\infty \sigma^* \left(\frac{r}{r_0}\right)^{s-1} \frac{\mathrm{d}r}{\mu r_0}, \quad \bar{\tau}(s) = \int_0^\infty \tau^* \left(\frac{r}{r_0}\right)^{s-1} \frac{dr}{\mu r_0}. \tag{21}$$

$\bar{u}$, $\bar{v}$ and $\bar{\sigma}$, $\bar{\tau}$ are dimensionless. Apply (16) and (17) so that

$$\left.\begin{aligned} \int_0^\infty \sigma^* J_0\left(\frac{pr}{b}\rho\right) r\,\mathrm{d}r &= \frac{\mu r_0^2}{2}\frac{1}{2\pi \mathrm{i}}\int \bar{\sigma}(1-s)\frac{\Gamma[\frac{1}{2}(1+s)]}{\Gamma[\frac{1}{2}(1-s)]}\left(\frac{pr_0\rho}{2b}\right)^{-s-1}\mathrm{d}s,\\ \int_0^\infty \tau^* J_1\left(\frac{pr}{b}\rho\right) r\,\mathrm{d}r &= \frac{\mu r_0^2}{2}\frac{1}{2\pi \mathrm{i}}\int \bar{\tau}(1-s)\frac{\Gamma(1+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\left(\frac{pr_0\rho}{2b}\right)^{-s-1}\mathrm{d}s. \end{aligned}\right\} \tag{22}$$

If (9), (18), and (20) are combined, it follows that

$$\left.\begin{aligned} -\bar{u}(s) &= \frac{\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\frac{1}{2\pi\mathrm{i}}\int \bar{\sigma}(1+s')\frac{\Gamma[\frac{1}{2}(1-s')]}{\Gamma[\frac{1}{2}(1+s')]}\bar{P}(s'-s)\left(\frac{pr_0}{2b}\right)^{s'-s}\mathrm{d}s'\\ &\quad+\frac{\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\frac{1}{2\pi\mathrm{i}}\int \bar{\tau}(1+s')\frac{\Gamma(1-\frac{1}{2}s')}{\Gamma(1+\frac{1}{2}s')}\bar{Q}(s'-s)\left(\frac{pr_0}{2b}\right)^{s'-s}\mathrm{d}s',\\ -\bar{v}(s) &= \frac{\Gamma[\frac{1}{2}(1+s)]}{\Gamma[\frac{1}{2}(3-s)]}\frac{1}{2\pi\mathrm{i}}\int \bar{\sigma}(1+s')\frac{\Gamma[\frac{1}{2}(1-s')]}{\Gamma[\frac{1}{2}(1+s')]}\bar{Q}(s'-s)\left(\frac{pr_0}{2b}\right)^{s'-s}\mathrm{d}s'\\ &\quad+\frac{\Gamma[\frac{1}{2}(1+s)]}{\Gamma[\frac{1}{2}(3-s)]}\frac{1}{2\pi\mathrm{i}}\int \bar{\tau}(1+s')\frac{\Gamma(1-\frac{1}{2}s')}{\Gamma(1+\frac{1}{2}s')}\bar{R}(s'-s)\left(\frac{pr_0}{2b}\right)^{s'-s}\mathrm{d}s', \end{aligned}\right\} \tag{23}$$

where the functions $\bar{P}(s)$, $\bar{Q}(s)$, and $\bar{R}(s)$ are given by

$$\begin{Bmatrix} \bar{P}(s) \\ \bar{Q}(s) \\ \bar{R}(s) \end{Bmatrix} = \int_0^\infty \mathrm{d}\rho \begin{Bmatrix} \frac{1}{2}\surd(k^{-2}+\rho^2) \\ \rho(\frac{1}{2}+\rho^2-\surd(1+\rho^2)\surd(k^{-2}+\rho^2)) \\ \frac{1}{2}\surd(1+\rho^2) \end{Bmatrix} \frac{\rho^s}{(1+2\rho^2)^2-4\rho^2\surd(k^{-2}+\rho^2)\surd(1+\rho^2)}. \tag{24}$$

The variable of integration s of (22) has been replaced by $-s'$ in the derivation of (23). The formulae (23) will be evaluated under the asumption that $-1 < \mathscr{R}\,(s'-s) < 0$ and $0 < \mathscr{R}\,s' < 1$. These limits are obtained from the following requirements. u and v can be expanded in powers of r, so that $0 < \mathscr{R}\,s < 1$. σ and τ can be expanded similarly so that $-1 < \mathscr{R}\,s' < 0$, because $\bar{\sigma}$ and $\bar{\tau}$ occur in (23) with the argument $1+s'$. The condition $-1 < \mathscr{R}\,(s'-s) < 0$ insures the convergence of the integrals (24). These three requirements are easily seen to be compatible with each other.

The integrands of (24) can be defined in the complex ρ plane with two cuts, one along the positive real axis and another connecting the branch points $\pm\mathrm{i}$ and $\pm\mathrm{i}k^{-1}$ in such a manner that it crosses the real axis at a negative point. The phase angles of ρ, α, and β are then defined to be zero on the upper part of the cut along the real positive axis. The integral from 0 to ∞ can be expressed in terms of a contour integral going around the cut along the real positive axis. This contour is then deformed into a contour which encircles counterclockwise the cut through the branch points $\pm\mathrm{i}$ and $\pm\mathrm{i}k^{-1}$. Finally a new variable of integration $\rho e^{-\mathrm{i}\pi}$ is introduced, which means a rotation of the complex ρ plane by 180° carrying cuts and contours along. In the new plane (again called ρ) the two cuts are one along the negative real axis and the other through the branch points $\pm\mathrm{i}$ and $\pm\mathrm{i}k^{-1}$ crossing the real axis at a positive point. To the right of these two cuts ρ, α, and β are now defined to have zero phase angles and the new contour of integration C encircles the branch points $\pm \mathrm{i}$ and $\pm \mathrm{i}k^{-1}$ counterclockwise (cf. the sketch of the complex ρ plane with the two cuts and the contour of integration C). After these manipulations and with the help of the formula

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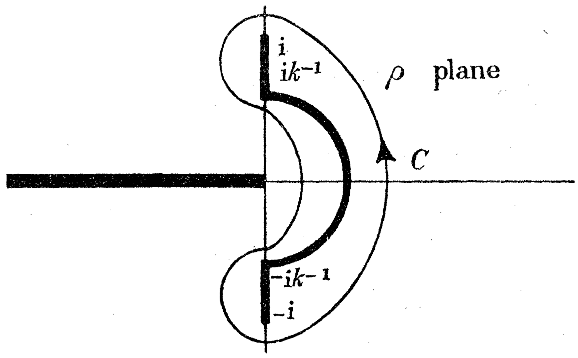

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad \Gamma(\tfrac{1}{2})\Gamma(s) = 2^{2s-1}\Gamma(\tfrac{1}{2}s)\Gamma[\tfrac{1}{2}(s+1)] \tag{25}$$

(cf. Whittaker & Watson 1952) the integrals (24) can be written as

$$\begin{Bmatrix} \bar{P}(s) \\ \bar{Q}(s) \\ \bar{R}(s) \end{Bmatrix} = \tfrac{1}{2}\begin{Bmatrix} \Gamma(-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\,P(s) \\ -\dfrac{\pi}{\sin(\frac{1}{2}\pi)\,s}\,Q(s), \\ \Gamma(-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\,R(s) \end{Bmatrix},$$

$$\begin{Bmatrix} P(s) \\ Q(s) \\ R(s) \end{Bmatrix} = \begin{Bmatrix} (1/\pi)\,\Gamma(1+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s) \\ \dfrac{1}{\cos(\frac{1}{2}\pi)\,s} \\ (1/\pi)\,\Gamma(1+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s) \end{Bmatrix} \frac{1}{2\pi \mathrm{i}}\int_C \frac{\rho^s\,\mathrm{d}\rho}{(1+2\rho^2)^2-4\rho^2\alpha\beta}\begin{Bmatrix} \tfrac{1}{2}\sqrt{(k^{-2}+\rho^2)} \\ \rho(\tfrac{1}{2}+\rho^2-\alpha\beta) \\ \tfrac{1}{2}\sqrt{(1+\rho^2)} \end{Bmatrix}. \tag{26}$$

The Γ functions factors have been chosen in such a way as to exhibit the poles of $\bar{P}(s)$, $\bar{Q}(s)$, and $\bar{R}(s)$, but make $P(s)$, $Q(s)$, and $R(s)$ regular throughout the s plane, as can easily be shown. Indeed, the contour integral can be evaluated if s is an integer by deforming C into a large circle with the centre at the origin of the s plane. This is permissible because for integer values of s the integrand of (26) can be defined in the complex ρ plane without the cut from $-\infty$ along the negative real axis to O. If s is an integer $\geqslant 0$, and the contour C is deformed into a large circle, it is necessary to expand the integrand for large values of ρ and retain the term in ρ^{-1}. If s is an negative integer, the integrand has a pole at $\rho = 0$ whose residue must be found before the contour C can be deformed into a large circle, which is then found to give no contribution to the integral. Finally, it is noticed that the expansions at infinity or at zero contain either only even or only odd powers of ρ, and that the contour integral (26) vanishes for integer values of s if there is no term ρ^{-1}. The corresponding zeros of the integral in (26) have been cancelled by the poles in the Γ-function factors appearing in (26). In this manner it is found, for instance, that

$$P(0) = R(0) = \frac{1}{(4\sqrt{\pi})\,(1-k^{-2})}, \quad Q(0) = -\frac{k^{-2}}{4(1-k^{-2})}. \tag{27}$$

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Because of (10), $\bar{\sigma}(s)$ and $\bar{\tau}(s)$ as defined by (21) are regular for $\mathscr{R}\, s > 0$. Assuming proper behaviour at infinity for the various functions in the integrand of (23), the integration over s' is performed by pulling the path of integration to the right. With the new functions

$$\left.\begin{aligned} \eta_1(s) &= -\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\bar{u}(s), \quad \eta_2(s) = -\frac{\Gamma(\frac{3}{2}-\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\bar{v}(s), \\ \zeta_1(s) &= \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\bar{\sigma}(1+s), \quad \zeta_2(s) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}s)}\bar{\tau}(1+s), \end{aligned}\right\} \tag{28}$$

the relations (23) become after some reduction

$$\begin{aligned} \frac{\eta_1(s)}{\Gamma(s)} = \frac{\cos\frac{1}{2}s\pi}{\sin\frac{1}{2}s\pi}\Big[\sum_{m=0}^{\infty}\frac{\zeta_2(s+2m)}{\Gamma(s+2m)}\left(\frac{pr_0}{b}\right)^{2m}Q(2m) - \sum_{m=0}^{\infty}\frac{\zeta_2(2m+2)}{\Gamma(2m+2)}\left(\frac{pr_0}{b}\right)^{2m+2-s}Q(2m+2-s)\Big] \\ + \sum_{m=0}^{\infty}\frac{\zeta_1(s+2m)\,\Gamma(m+\frac{1}{2})}{\Gamma(s+2m)\,\Gamma(m+1)}\left(\frac{pr_0}{b}\right)^{2m}P(2m) \\ - \sum_{m=0}^{\infty}\frac{\zeta_1(2m+1)\,\Gamma(m+1-\frac{1}{2}s)}{\Gamma(2m+1)\,\Gamma(m+\frac{3}{2}-\frac{1}{2}s)}\left(\frac{pr_0}{b}\right)^{2m+1-s}P(2m+1-s), \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\eta_2(s)}{\Gamma(s)} = \frac{\sin(\frac{1}{2}s)\,\pi}{\cos(\frac{1}{2}s)\,\pi}\Big[\sum_{m=0}^{\infty}\frac{\zeta_1(s+2m)}{\Gamma(s+2m)}\left(\frac{pr_0}{b}\right)^{2m}Q(2m) - \sum_{m=0}^{\infty}\frac{\zeta_1(2m+1)}{\Gamma(2m+1)}\left(\frac{pr_0}{b}\right)^{2m+1-s}(Q2m+1-s)\Big] \\ + \sum_{m=0}^{\infty}\frac{\zeta_2(s+2m)\,\Gamma(m+\frac{1}{2})}{\Gamma(s+2m)\,\Gamma(m+1)}\left(\frac{pr_0}{b}\right)^{2m}R(2m) \\ - \sum_{m=0}^{\infty}\frac{\zeta_2(2m+2)\,\Gamma(m+\frac{3}{2}-\frac{1}{2}s)}{\Gamma(2m+2)\,\Gamma(m+2-\frac{1}{2}s)}\left(\frac{pr_0}{b}\right)^{2m+2-s}R(2m+2-s). \end{aligned} \tag{30}$$

From the asymptotic formulae for $P(s)$, $Q(s)$, and $R(s)$ for large s and from the assumed algebraic behaviour of $\zeta_1(s)$ and $\zeta_2(s)$ it follows that (29) and (30) converge for all values of s and p.

The boundary conditions (10), (11), and (12) are equivalent to the following requirements

(1) $\zeta_1(s)$ is regular for $\mathscr{R}\, s > -1$ except for a first-order pole at $s = 0$;

(2) $\zeta_2(s)$ is regular for $\mathscr{R}\, s > -1$;

(3) $\eta_1(s)$ is regular for $\mathscr{R}\, s < \delta$, where δ is a sufficiently small positive number, except a first-order pole at $s = 0$ with residue equal to $(-1/\sqrt{\pi})\,h(p)$;

(4) the function

$$D\frac{pr_0}{b}\eta_2(s) - (1-D)\frac{\sin\frac{1}{2}s\pi}{\cos\frac{1}{2}s\pi}\tfrac{1}{2}(s-1)\,\zeta_2(s-1) \tag{31}$$

is regular for $\mathscr{R}\, s < \delta$.

If it is assumed that σ^* and τ^* can be expanded in powers of r/r_0, then $\zeta_1(s)$ and $\zeta_2(s)$ have only first-order poles, which are located at the negative integers. In particular, $\zeta_1(s)$ has poles only at the negative even integers (including of course 0) and $\zeta_2(s)$ has poles only at the negative odd integers.

In the case $D = 0$, it follows directly from (12) and (10) that $\tau^* = 0$ and therefore $\zeta_2(s) = 0$. The right-hand side of (29) reduces to the last two lines, which has to vanish for $s = -2n$ (where n is a positive integer) and equals $-(1/\sqrt{\pi})\,h(p)$ for $s = 0$. If the residue of $\zeta_1(s)$ at $s = -2n$ is called $\phi_n(y)$ with $y = pr_0/b$ equation (29) shows that an expansion of

$\phi_n(y)$ in powers of y starts with a term $y^{\nu+2n}$ if the expansion of $\phi_0(y)$ starts with a term y^ν. It follows from (13) that an expansion of σ^* in powers of y starts with a constant term if the weight Mg of the disk is taken into account, and with a linear term if the weight is neglected, so that only the velocity term pw remains in (13). Therefore $\zeta_1(s)$ can be written as

$$\zeta_1(s) = \sum_{n=0}^{\infty} \frac{\phi_n(y)}{s+2n}, \quad \phi_n(y) = y^{\nu+2n}(A_n + B_n y + \ldots), \tag{32}$$

where $\nu = 0$ if the term Mg in (13) is kept, otherwise $\nu = 1$.

The coefficients in the power series $\phi_n(y)$ can be determined successively, i.e. without solving simultaneous equations. This necessitates a detailed writing of the condition (13) and the equation (29) for $s = 0, -2, -4, \ldots$ in terms of these coefficients, and leads to straightforward but very lengthy computations. Therefore a different approach has been used in the special investigation of the case $D = 0$. From combining (28), (21), and the inversion formula (15) it is found that

$$\sigma^* = \mu \sum_{m=0}^{\infty} f_m(y) \left(1 - \frac{r^2}{r_0^2}\right)^{m-\frac{1}{2}}, \tag{33}$$

with
$$f_m(y) = \frac{(-1)^m}{\Gamma(m+\frac{1}{2})} \sum_{n=m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \phi_n(y).$$

Therefore an expansion of the functions $f_m(y)$ again starts with a term y^{2m} or y^{2m+1} according to whether or not the weight term Mg is kept in (13).

The cases $D > 0$, i.e. the cases of viscous friction and perfect adherence, cannot be treated in this simple manner and no attempt will be made to derive a formula which would be similar to (33). As a formula like (33) is necessary in order to apply the method which was used in the special investigation of the case $D = 0$, i.e. the case of perfect lubrication, no quantitative results can be stated for the cases $D > 0$, unless these cases can be shown to be close to the case $D = 0$, at least under some additional assumptions concerning the constant k in (1). Although there is only the physical intuition to support this view, it seems that among the cases $D > 0$ the special case $D = 1$ (perfect adherence) is the furthest removed from the special case $D = 0$. It also has the advantage of not introducing a new parameter in the calculation, such as the coefficient D of viscous friction in (13). It is therefore considered as the most interesting for an investigation of the relation between the cases $D > 0$ and the case $D = 0$.

3. The case of perfect adherence

A partial fraction expansion

$$\zeta_1(s) = \sum_{n=0}^{\infty} \frac{\phi_n(y)}{s+2n}, \quad \zeta_2(s) = \sum_{n=0}^{\infty} \frac{\psi_n(y)}{s+2n+1} \tag{34}$$

is assumed in analogy to the case $D = 0$. The two expansions (34) are equivalent to expanding σ directly in terms of $(1-r^2/r_0^2)^{n-\frac{1}{2}}$, and τ in terms of r times $(1-r^2/r_0^2)^{n-\frac{1}{2}}$ with $n = 0, 1, 2, \ldots$. But compared to the expanison (33) the information is lost that $\phi_n(y)$ or $\psi_n(y)$ in these expansions (34) have a lowest power y^{2n} or y^{2n+1} in their dependence on y. A set of equations for the residues $\phi_n(y)$ and $\psi_n(y)$ results, if one puts $s = -2l$ in (29) and $s = -2l-1$ in (30) with the integers l ranging from 0 to ∞. The left-hand sides of (29) and (30) vanish for these values of s, except for $s = 0$ in (29) where it equals $-(1/\sqrt{\pi})\,h(p)$. The equation of motion (13) for the rigid disk follows from (15), (21) and (28) as

$$M[p^2h(p)-pw] = 2\sqrt{\pi}\,\mu r_0^2 \sum_{n=0}^{\infty} \frac{\phi_n(y)}{2n+1} + Mg. \tag{35}$$

The functions $\phi_n(y)$ and $\psi_n(y)$ will be expanded around the point $p = 0$ (or $y = 0$), but it is not possible to have a simple power series, because terms containing $\log y$ cannot be avoided. Such $\log y$ terms arise from the first part of (29) and (30), where it is necessary to consider the limits $s \to -2l$ and $s \to -2l+1$ in order to cancel the zeros of the denominator. The terms in the expansion will be ordered according to increasing exponents in the powers of y, so that

$$\phi_n(y) = \sum_{m=0}^{\infty} y^m \phi_n^{(m)}(y), \quad \psi_n(y) = \sum_{m=0}^{\infty} y^m \psi_n^{(m)}(y), \tag{36}$$

where $$\lim_{y\to 0} \phi_n^{(m)}(y) \neq 0, \quad \lim_{y\to 0} \psi_n^{(m)}(y) \neq 0,$$

but $$\lim_{y\to 0} y\phi_n^{(m)}(y) = 0, \quad \lim_{y\to 0} y\psi_n^{(m)}(y) = 0.$$

In the same way $h(p)$ is expanded as

$$h(p) = \sum_{m=0}^{\infty} y^m h^{(m)}(y), \tag{37}$$

with $$\lim_{y\to 0} h^{(m)}(y) \neq 0,$$

but $$\lim_{y\to 0} y h^{(m)}(y) = 0.$$

If the weight term Mg is taken into consideration in (35), the expansions (36) and (37) start indeed with a term $m = 0$. Otherwise they start with a linear term.

For the lowest term in the expansion (36) the equations (29) for $s = -2l$ and (30) for $s = -2l-1$ reduce to

$$\begin{aligned} -\frac{4(1-k^{-2})}{\sqrt{\pi}}\,h^{(0)}\,\delta_{0l} &= \phi_l^{(0)} - \frac{2}{\pi k^2} \sum_{n=0}^{\infty} \frac{\psi_n^{(0)}}{-2l+2n+1}, \\ 0 &= \psi_l^{(0)} + \frac{2}{\pi k^2} \sum_{n=0}^{\infty} \frac{\phi_n^{(0)}}{-2l-1+2n}, \end{aligned} \tag{38}$$

where (27) has been used. (δ_{mn} is the Kronecker symbol; $\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$.) The next term in y gives

$$\begin{aligned} -\frac{4(1-k^{-2})}{\sqrt{\pi}}\,h^{(1)}\,\delta_{0l} &= \phi_l^{(1)} - \frac{2}{\pi k^2} \sum_{n=0}^{\infty} \frac{\psi_n^{(1)}}{-2l+2n+1} - \frac{8(1-k^{-2})}{\sqrt{\pi}}\,P(1)\,\delta_{0l} \sum_{n=0}^{\infty} \frac{\phi_n^{(0)}}{2n+1}, \\ 0 &= \psi_l^{(1)} + \frac{2}{\pi k^2} \sum_{n=0}^{\infty} \frac{\phi_n^{(1)}}{-2l-1+2n}. \end{aligned} \tag{39}$$

Because of (35) the $\phi_n^{(0)}$ and $\phi_n^{(1)}$ have to satisfy also

$$\begin{aligned} 2\sqrt{2}\,\mu r_0^2 \sum_{n=0}^{\infty} \frac{\phi_n^{(0)}}{2n+1} + Mg &= 0, \\ 2\sqrt{2}\,\mu r_0^3 \sum_{n=0}^{\infty} \frac{\phi_n^{(1)}}{2n+1} + Mwb &= 0. \end{aligned} \tag{40}$$

Similar equations are obtained for the higher terms in the expansions of $\phi_n(y)$ and $\psi_n(y)$.

The equations (38), (39), and the analogous equations for $\phi_n^{(2)}$, $\psi_n^{(2)}$, etc., always have the form

$$\alpha_l - k^{-2}\sum_{n=0}^{\infty} C_{nl}\beta_n = a_l,$$

$$\beta_l - k^{-2}\sum_{n=0}^{\infty} C_{ln}\alpha_n = b_l, \tag{41}$$

$$C_{ln} = \frac{1}{\pi}\frac{1}{l-n+\frac{1}{2}} \quad (1 \geqslant 0, \quad n \geqslant 0). \tag{42}$$

It should be noted that the equations for $\phi_n^{(2)}$ and $\psi_n^{(2)}$ have, in the notation of (41), all $b_l \neq 0$ even if $k^{-2} = 0$. The cases $D = 0$ and $D = 1$ do not coincide even if $k^{-2} = 0$ for that term and all the higher terms. The quantities a_l and b_l are known from equations (29), (30), and (31) except α_0, which is obtained from the additional condition

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2n+1} = c_0, \tag{43}$$

where c_0 is known from (35). As Poisson's ratio is limited by the condition $\nu > -1$, the parameter k^2 has a lower limit which is given according to expression (1) by the inequality $k^2 > \frac{4}{3}$. It is therefore natural to expand the solution of the linear equations (41) in powers of k^{-2}.

With the help of matrix calculus the equations (41) can be written as

$$\left.\begin{aligned} \bar{\alpha} &= \bar{a} + k^{-2}C'\bar{\beta}, \\ \bar{\beta} &= \bar{b} + k^{-2}C\bar{\alpha}. \end{aligned}\right\} \tag{44}$$

$\bar{\alpha}$ is the vector $(\alpha_0, \alpha_1, \ldots)$ etc., C is the matrix (C_{ln}) and C' the transposed of C, i.e. C' is obtained from C by switching the indices of its elements. From (44) it follows that

$$\bar{\alpha} = \bar{a} + k^{-2}C'\bar{b} + k^{-4}C'C\bar{\alpha},$$

$$\bar{\beta} = \bar{b} + k^{-2}C\bar{a} + k^{-4}CC'\bar{\beta}.$$

Therefore (with N being any positive integer)

$$\left.\begin{aligned} \bar{\alpha} &= \left[\sum_{m=0}^{N-1} k^{-4m}(C'C)^m\right](\bar{a} + k^{-2}C'\bar{b}) + k^{-4N}(C'C)^N\bar{\alpha}, \\ \bar{\beta} &= \left[\sum_{m=0}^{N-1} k^{-4m}(CC')^m\right](\bar{b} + k^{-2}C\bar{a}) + k^{-4N}(CC')^N\bar{\beta}. \end{aligned}\right\} \tag{45}$$

It is shown in appendix A that

$$(CC')^{N+1}|_{nl} < \delta_{nl} + \frac{2}{(\pi^2+2)\sqrt{(n+1)}\sqrt{(l+1)}}\left[\frac{3\pi^2+2}{2\pi^2}\right]^{N+1} \quad \text{for} \quad N \geqslant 0. \tag{46}$$

With the abbreviation

$$d_l = \sum_{n=0}^{\infty} |C_{ln}| \frac{1}{\sqrt{(n+1)}},$$

it follows also that

$$(C'C)^{N+2}|_{nl} < C'C|_{nl} + \frac{2d_l d_n}{\pi^2+2}\left[\frac{3\pi^2+2}{2\pi^2}\right]^{N+1}. \tag{47}$$

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The expressions (45) for $\bar{\alpha}$ and $\bar{\beta}$ converge for N going to ∞ to a solution of (44) provided

$$k^2 > \sqrt{\left(\frac{3\pi^2+2}{2\pi^2}\right)} = 1{\cdot}265433 \tag{48}$$

and the series

$$\sum_{n=0}^{\infty} d_n|a_n+k^{-2}(C'\bar{b})_n|,$$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+1)}}\,|b_n+k^{-2}(C\bar{a})_n| \tag{49}$$

converge. It can easily be shown that for large n, d_n behaves as $(n+1)^{-\frac{1}{2}}\log(n+1)$. Condition (48) is actually weaker than $k^2 > \frac{4}{3}$ which holds as a consequence of (1).

Consider now in particular the static limit which is obtained by letting y go to zero. The solution results from (38) and the first part of (40). As a first approximation, one may take

$$\left.\begin{aligned} \phi_l^{(0)} &= -\frac{4(1-k^{-2})}{\sqrt{\pi}}\,h^{(0)}\delta_{0l},\\ \psi_l^{(0)} &= -\frac{4(1-k^{-2})}{\sqrt{\pi}}\,h^{(0)}k^{-2}C_{l0}.\end{aligned}\right\} \tag{50}$$

With the help of (40) and of the formula

$$\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})\,(s+2n+1)} = \frac{\psi[\frac{1}{2}(1+s)]-\psi(\frac{1}{2})}{s}, \tag{51}$$

where ψ is the logarithmic derivative of the Γ function, it follows that

$$\begin{aligned} \zeta_1(s) &= -\frac{Mg}{2\sqrt{\pi}\mu r_0^2}\,\frac{1}{s},\\ \zeta_2(s) &= -\frac{Mg}{2\sqrt{\pi}\mu r_0^2}\,\frac{\psi[\frac{1}{2}(1+s)]-\psi(\frac{1}{2})}{\pi k^2 s}.\end{aligned} \tag{52}$$

The corresponding (transformed) normal and tangential displacements are directly obtained from (29) and (30) by inserting (52) and letting $y = pr_0/b$ go to zero. With the help of (28) it results that

$$\left.\begin{aligned} \bar{\sigma}(s) &= -\frac{Mg}{4\sqrt{\pi}\mu r_0^2}\,\frac{\Gamma(\frac{1}{2}s)}{\Gamma[\frac{1}{2}(s+1)]},\\ \bar{\tau}(s) &= -\frac{Mg}{4\sqrt{\pi}\mu r_0^2}\,\frac{\Gamma[\frac{1}{2}(s-1)]}{\pi k^2\,\Gamma(\frac{1}{2}s)}\,[\psi(\tfrac{1}{2}s)-\psi(\tfrac{1}{2})];\end{aligned}\right\} \tag{53}$$

$$\bar{u}(s) = \frac{Mg}{8\sqrt{\pi}\mu r_0^2\,(1-k^{-2})}\left\{\frac{\Gamma[\frac{1}{2}(1-s)]}{s\,\Gamma(1-\frac{1}{2}s)} - \frac{\Gamma(\frac{1}{2}s-1)}{2\pi k^4\,\Gamma[\frac{1}{2}(1+s)]}\,[\psi[\tfrac{1}{2}(1+s)]-\psi(\tfrac{1}{2})]\right\}; \tag{54}$$

$$\bar{v}(s) = -\frac{Mg}{8\sqrt{\pi}\mu r_0^2\,(1-k^{-2})}\,\frac{\Gamma(-\frac{1}{2}s)}{2k^2\,\Gamma[\frac{1}{2}(3-s)]}\left\{\frac{\sin\frac{1}{2}s\pi}{\cos\frac{1}{2}s\pi} - \frac{1}{\pi}\psi[\tfrac{1}{2}(1+s)]-\psi(\tfrac{1}{2})\right\}. \tag{55}$$

From (15), (20) and (21) it follows that

$$\left.\begin{aligned} \sigma &= -\frac{Mg}{2\pi r_0^2}\left(1-\frac{r^2}{r_0^2}\right)^{-\frac{1}{2}} && \text{for} \quad r < r_0,\\ &\quad 0 && \text{for} \quad r > r_0;\end{aligned}\right\} \tag{56}$$

$$\tau = \begin{cases} \dfrac{Mg}{2\pi r_0^2}\left(1-\dfrac{r^2}{r_0^2}\right)^{-\frac{1}{2}} \dfrac{r_0}{\pi k^2 r}\log\left(1-\dfrac{r^2}{r_0^2}\right) & \text{for} \quad r<r_0, \\ 0 & \text{for} \quad r>r_0; \end{cases} \tag{57}$$

$$u = \frac{Mg}{8\mu r_0(1-k^{-2})}\left[1+\frac{4}{\pi^2 k^4} f\left(\frac{r}{r_0}\right)\right] \quad \text{for} \quad r<r_0;$$

$$v = 0 \qquad \text{for} \quad r<r_0. \tag{58}$$

The function

$$f(x) = x^{-2}(1+\surd(1-x^2)\log\surd(1-x^2)-\surd(1-x^2)) \tag{59}$$

increases monotonically from 0 at $x = 0$ to 1 at $x = 1$. This latter value is reached with vertical slope. The star indicating the 'p-multiplied' Laplace transform in (56), (57), and (58) has been omitted, because in the limit of p going to zero this transform becomes equal to its original taken in the limit of t going to infinity, i.e. the static limit.

It should be noted that the stresses and displacements as given in (56), (57), and (58) are in equilibrium, if they are applied to the plane boundary of a semi-infinite elastic solid. But one among the postulated boundary conditions, namely that the normal displacement u is constant for $r<r_0$, is not satisfied, where the other boundary conditions are fulfilled. Analogous solutions for the higher powers in pr_0/b can be obtained in a similar way; e.g. the linear terms in pr_0/b result from (39) and the second part of (40).

4. The normal displacement due to certain stress distributions

If the case of perfect lubrication, i.e. $D = 0$ in (12) or $\tau^* = 0$, is investigated, the formula (33) suggests as a good approximation for the normal stress σ^* due to the impinging rigid disk in the contact area $x<1$

$$\sigma^* = \mu\sum_{n=0}^{N} f_n(y)\,(1-x^2)^{n-\frac{1}{2}} \quad \text{with} \quad x = \frac{r}{r_0}, \quad y = \frac{pr_0}{b}, \quad \text{and} \quad \sigma^* = 0 \quad \text{for} \quad x>1. \tag{60}$$

In the case of perfect adherence, a quite similar treatment of the problem could be given if in addition to the expansion (60) for σ^*, the shear stress τ^* in the contract area is expanded in terms of r times $(1-r^2/r_0^2)^{n-\frac{1}{2}}$ with $n = 0, 1, \ldots$. But the algebra to be performed would become more complicated. Condition (11) cannot be satisfied exactly, but the approximation should be good for small values of y, which means for slow motions of the rigid disk. The most logical method to compute u^* corresponding to (60) would be to combine (19), (20), (21), (28), and (29) using the fact that $\tau^* = 0$. It has been thought desirable, however, to obtain this result independently of §2, because the condition $\tau^* = 0$ introduces some simplifications in the argument. With the help of the formula

$$\int_0^1 (1-x^2)^{n-\frac{1}{2}} J_0(\xi x)\,x\,\mathrm{d}x = 2^{n-\frac{1}{2}}\Gamma(n+\tfrac{1}{2})\,\xi^{-n-\frac{1}{2}} J_{n+\frac{1}{2}}(\xi), \tag{61}$$

it follows from (8) and (9), with the boundary conditions $\sigma^* = 0$ for $x>1$ and $\tau^* = 0$ for $x>0$, that

$$u^* = -r_0\sum_{n=0}^{N}\Gamma(n+\tfrac{1}{2})f_n(y)\int_0^\infty \frac{\alpha\,\mathrm{d}\rho}{(1+2\rho^2)^2-4\rho^2\alpha\beta}\left(\frac{\rho y}{2}\right)^{\frac{1}{2}-n} J_{n+\frac{1}{2}}(y\rho)\,J_0(xy\rho). \tag{62}$$

The integral over ρ has now to be brought into a form which allows its numerical evalution.

If the power series for $J_{n+\frac{1}{2}}$ and J_0 are multiplied term by term, one obtains

$$\left(\frac{\rho y}{2}\right)^{-n-\frac{1}{2}} J_{n+\frac{1}{2}}(y\rho)\, J_0(xy\rho) = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}}{\Gamma(\lambda+1)\,\Gamma(n+\frac{3}{2}+\lambda)} \left(\frac{\rho y}{2}\right)^{2\lambda} F(-\lambda,\, -n-\lambda-\tfrac{1}{2};\, 1;\, x^2). \tag{63}$$

The hypergeometric function $F(-\lambda,\, -n-\lambda-\frac{1}{2};\, 1;\, x^2)$ can be expressed differently by the use of the formula

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right). \tag{64}$$

With the new variable ξ defined by

$$x = \tanh \xi \quad \text{or} \quad \frac{1+x^2}{1-x^2} = \cosh 2\xi \tag{65}$$

and the integral representation

$$F(-\lambda, n+\tfrac{3}{2}+\lambda; 1; \tfrac{1}{2}(1-\cosh 2\xi))$$
$$= \frac{2}{\pi}\int_{e^{-\xi}}^{e^{\xi}} \mathscr{R}\left[\frac{\sqrt{\{(\eta+e^{-\xi})\,(e^{\xi}+\eta)\}}-\mathrm{i}\sqrt{\{(\eta-\mathrm{e}^{-\xi})\,(e^{\xi}-\eta)\}}}{2\cosh\xi}\right]^{2n+1} \frac{\eta^{2\lambda+1}\,\mathrm{d}\eta}{\sqrt{\{(\mathrm{e}^{2\xi}-\eta^2)\,(\eta^2-\mathrm{e}^{-2\xi})\}}} \tag{66}$$

(cf. Appendix B), the expression (62) for u^* becomes

$$u^* = -\frac{2r_0}{\pi}\cosh\xi \sum_{n=0}^{N} \Gamma(n+\tfrac{1}{2}) f_n(y) \int_{e^{-\xi}}^{e^{\xi}} \frac{\mathrm{d}\eta}{\sqrt{\{(e^{2\xi}-\eta^2)\,(\eta^2-\mathrm{e}^{-2\xi})\}}}$$
$$\times \mathscr{R}\left(\frac{\sqrt{\{\ldots\}}-\mathrm{i}\sqrt{\{\ldots\}}}{2\cosh\xi}\right)^{2n+1} \int_0^{\infty} \frac{\alpha\,\mathrm{d}\rho}{(1+2\rho^2)^2-4\rho^2\alpha\beta} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}}{\Gamma(\lambda+1)\,\Gamma(n+\frac{3}{2}+\lambda)} \left(\frac{\rho\eta y}{2\cosh\xi}\right)^{2\lambda+1}. \tag{67}$$

The roots in $\mathscr{R}(\quad)^{2n+1}$ are the same as in (66). The interchange of the ρ integration with the η integration can be justified because

$$\sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}\zeta^{2\lambda}}{\Gamma(\lambda+1)\,\Gamma(n+\frac{3}{2}+\lambda)} = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{1}{2\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^n \frac{\sin 2\zeta}{\zeta}, \tag{68}$$

so that both the ρ integration and the η integration are absolutely convergent. This can be verified at once using the power series for $\sin 2\zeta$. The power series (68) is now written as an integral of Barnes's type.

$$\sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}\xi^{2\lambda}}{\Gamma(\lambda+1)\,\Gamma(n+\frac{3}{2}+\lambda)} = \frac{1}{2\pi\mathrm{i}}\int \frac{\Gamma(-s)}{\Gamma(n+\frac{3}{2}+s)}\,\zeta^{2s}\,\mathrm{d}s, \tag{69}$$

where the path of integration is parallel to the imaginary axis of the complex s plane with a negative real part. After the s integration has been interchanged with the ρ integration, the second line of (67) becomes, by the use of the formulas (24), (25), and (26),

$$\frac{1}{2\pi\mathrm{i}}\int \frac{2\pi^2}{\sin s\pi}\,\frac{P(2s+1)}{\Gamma(n+\frac{3}{2}+s)\,\Gamma(\frac{3}{2}+s)}\left(\frac{\eta y}{2\cosh\xi}\right)^{2s+1}\mathrm{d}s, \tag{70}$$

where the integration has to be restricted by the condition $-1 < \mathscr{R}\, s < -\frac{1}{2}$ in order to make the interchange of integration legitimate. The expression (70) is evaluated by pulling the s integration to the right across the poles of the integrand at $s = -\frac{1}{2}$, 0, $+\frac{1}{2}$, 1, etc., giving

$$\pi \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda} P(\lambda)}{\Gamma(1+\frac{1}{2}\lambda)\,\Gamma(n+1+\frac{1}{2}\lambda)}\left(\frac{\eta y}{2\cosh\xi}\right)^{\lambda}. \tag{71}$$

If this is inserted into the second line of (67) and formulae (66) and (64) are used to evaluate the integral over η, one finally obtains

$$u^* = -\pi r_0 \sum_{n=0}^{N} \Gamma(n+\tfrac{1}{2}) f_n(y) \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda P(\lambda)}{\Gamma(1+\frac{1}{2}\lambda)\,\Gamma(n+1+\frac{1}{2}\lambda)} F(\tfrac{1}{2}-\tfrac{1}{2}\lambda, -n-\tfrac{1}{2}\lambda; 1; x^2)(\tfrac{1}{2}y)^{\frac{\lambda}{2}}. \quad (72)$$

Whereas $P(\lambda)$ depends only on the constant k^2 of (1), the hypergeometric functions depend only on n, i.e. the assumed stress distribution in (60).

Formula (72) gives an expression of u^* in powers of y with increasing exponents. It is possible to obtain from (70) an expansion of u^* in powers of y with decreasing exponents. To this end insert (26) into (70), interchange the ρ integration with the s integration, and insert the result into the second line of (67). The result is

$$\begin{aligned} u^* = &-\frac{2r_0}{\pi}\cosh\xi \sum_{n=0}^{N}\Gamma(n+\tfrac{1}{2}) f_n(y) \int_{e^{-\xi}}^{e^{\xi}} \frac{d\eta}{\sqrt{\{(e^{2\xi}-\eta^2)(\eta^2-e^{-2\xi})\}}} \\ &\times \mathscr{R}\left(\frac{\sqrt{\{\dots\}}-i\sqrt{\{\dots\}}}{2\cosh\xi}\right)^{2n+1} \frac{1}{2\pi i}\int_C \frac{\frac{1}{2}\alpha\, d\rho}{(1+2\rho^2)^2-4\rho^2\alpha\beta} \\ &\times \frac{1}{2\pi i}\int \frac{2\pi\Gamma(-s)}{\sin 2\pi s\,\Gamma(n+\frac{3}{2}+s)}\left(\frac{\rho\eta y}{2\cosh\xi}\right)^{2s+1} ds, \end{aligned} \quad (73)$$

where the s integration is still restricted to $-1 < \mathscr{R}s < -\frac{1}{2}$, and the contour C encircles the points $\pm i$ and $\pm ik^{-1}$ counterclockwise, but does not cross the negative real axis including the origin of the ρ plane. The absolute values of ρ along C has therefore a finite upper bound and a non-zero lower bound, and its phase angles are bounded by $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. The s integration is therefore convergent, if the phase angle of y is bounded by $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, and an asymptotic expansion for large y is obtained under this condition, if the path of the s integration is pulled to the left across the poles of the integrand at -1, $-\frac{3}{2}$, etc. The first term in the asymptotic expansion of u^* for large y becomes with the help of (64) and (66)

$$u^* \sim -\frac{r_0}{k}\sum_{n=0}^{N}\frac{f_n(\infty)}{y}(1-x^2)^{n-\frac{1}{2}}, \quad (74)$$

provided $f_n(y)$ has the limit $f_n(\infty)$ as y increases indefinitely. Now the limit of σ^* as y goes to infinity equals σ at $t=0$. Also the limit of pu^* as y goes to infinity equals $\partial u/\partial t$ at $t=0$. If follows therefore from (60) and (74) that

$$\lim_{t\to 0}\frac{1}{b}\frac{\partial u}{\partial t} = -\frac{1}{k}\lim_{t\to 0}\frac{1}{\mu}\sigma. \quad (75)$$

This last equation can actually be derived in a simpler manner.

5. Determination of the stress distributions

The boundary conditions (11) and (13) cannot be satisfied rigorously with the assumption (60). However, a scheme will be proposed which determines the functions $f_n(y)$ in (60) in such a way as to approximate the conditions (11) and (13). The main justification for this scheme is that it can be reasonably expected to achieve this approximation, and that it leads to relatively simple formulas which can be handled by a high-speed computer. But no general argument has been found which would distinguish this scheme from similar ones as being the best in some sense.

The boundary conditions (10) and (12) are satisfied with the assumption (60) and $\tau^* = 0$. The properties of u^* are especially important at the rim of the rigid disk, i.e. for x close to 1. Therefore u^* is expressed in terms of $\xi = \surd(1-x^2)$ rather than x itself. u^* is then expanded in terms of the even Legendre polynomials $P_{2\nu}(\xi)$. The coefficient C_ν of $P_{2\nu}(\xi)$ is given by $(4\nu+1)\int_0^1 u^* P_{2\nu}(\xi)\,d\xi$. In particular $C_0 = \int_0^1 u^*\,d\xi$, because $P_0(\xi) = 1$. Also, the mean square deviation of u^* from C_0, i.e. $\int_0^1 (u^*-C_0)^2\,d\xi$, is equal to

$$\sum_{\nu=1}^{\infty} \frac{C_\nu^2}{4\nu+1}.$$

If the condition (11) were satisfied, this last sum would vanish. With the assumption (60) one can at least choose $f_n(y)$ such that $C_\nu = 0$ for $\nu = 1, \ldots, N$. This leaves the possibility to impose one further condition on u^*, namely (13), where $h(p)$ is now replaced by the average $2\int_0^1 u^* x\,dx$.

Condition (11) is now replaced by

$$\int_0^1 u^* P_{2\nu}[\surd(1-x^2)]\frac{x\,dx}{\surd(1-x^2)} = 0 \quad \text{for} \quad \nu = 1, \ldots, N; \tag{76}$$

i.e. the displacement u^* which results from the stress distribution (60) is required to be orthogonal to the even Legendre polynomials of order greater than zero with the argument $\surd(1-x^2)$ and a weight function $1/\surd(1-x^2)$. Condition (13) is replaced by

$$M\left[p^2\int_0^1 u^* 2x\,dx - pw\right] = 2\pi r_0^2\int_0^1 \sigma^* x\,dx + Mg, \tag{77}$$

i.e. the equation of motion and the initial conditions are postulated for the average of the displacement which results from the stress distribution (60).

Because of the strong convergence of (72) the integrations (76) and (77) can be performed in each term of the series (72). According to the results of appendix C the conditions (76) and (77) can be written as

$$\sum_{n=0}^{N} 2^n\Gamma(n+\tfrac{1}{2})f_n(y)\left[y^2G_n(y) + \frac{\surd(2\pi)\,\mu r_0^3}{Mb^2}\,\frac{1}{1.3\ldots(2n+1)}\right] = -\frac{wy}{\surd 2b} - \frac{gr_0}{\surd 2b^2}, \tag{78}$$

$$\sum_{n=0}^{N} 2^n\Gamma(n+\tfrac{1}{2})f_n(y)\,H_{\nu n}(y) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, N, \tag{79}$$

$$G_n(y) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+\frac{3}{4})\,\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+\frac{5}{4})\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(n+1+\frac{1}{2}\lambda)\,\Gamma(n+2+\frac{1}{2}\lambda)\,\Gamma(\frac{1}{2}\lambda+\frac{3}{2})}\,\frac{(-1)^\lambda P(\lambda)}{\Gamma(\lambda+1)}\,(2y)^\lambda, \tag{80}$$

$$H_{\nu n}(y) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+1)\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2})\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(1+n-\nu+\frac{1}{2}\lambda)\,\Gamma(n+\nu+\frac{1}{2}\lambda+\frac{3}{2})\,\Gamma(\frac{1}{2}-\nu+\frac{1}{2}\lambda)\,\Gamma(1+\nu+\frac{1}{2}\lambda)}\,\frac{(-1)^\lambda P(\lambda)}{\Gamma(\lambda+1)}\,(2y)^\lambda, \tag{81}$$

where coefficients with a Γ function of negative integer argument in the denominator are supposed to vanish. As $P(\lambda)$ behaves like an exponential of λ (cf. the expression of $P(s)$ in terms of hypergeometric functions as derived in § 8, cf. also Watson (1918)), the series (80) and (81) converge like the power series for exp. $(2y)$.

The asymptotic expansions of $G_n(y)$ and $H_{\nu n}(y)$ for large y are determined in appendix D. For real positive y the first term in the expansion is

$$G_n(y) \sim \frac{\sqrt{2}}{1.3\ldots(2n+1)\sqrt{\pi k}}\frac{1}{y}; \tag{82}$$

$$H_{\nu 0}(y) \sim \frac{\log 2y}{k\pi}\frac{1}{y}; \tag{83}$$

$$H_{\nu n}(y) \sim \frac{\Gamma(\nu+1)\,2^{-n}\Gamma(n)}{(-1)^\nu k\pi\Gamma(\nu+\frac{1}{2})\,\Gamma(n-\nu+\frac{1}{2})\,\Gamma(n+\nu+1)}\frac{1}{y} \quad \text{for} \quad n>0. \tag{84}$$

If these expressions are inserted into (78) and (79), the limits $f_n(\infty)$ can be obtained by solving the system of linear equations. From (60) and (75) it follows then that

$$\begin{aligned}\lim_{t\to 0}\frac{1}{w}\frac{\partial u}{\partial t} &= \tfrac{3}{2}(1-x^2)^{\frac{1}{2}} \quad \text{for} \quad N=1,\\ &= \tfrac{5}{2}(1-x^2)^{\frac{1}{2}}-\tfrac{5}{3}(1-x^2)^{\frac{3}{2}} \quad \text{for} \quad N=2,\\ &= \tfrac{7}{2}(1-x^2)^{\frac{1}{2}}-\tfrac{35}{6}(1-x^2)^{\frac{3}{2}}+\tfrac{7}{2}(1-x^2)^{\frac{5}{2}} \quad \text{for} \quad N=3,\\ &= \tfrac{9}{2}(1-x^2)^{\frac{1}{2}}-\tfrac{27}{2}(1-x^2)^{\frac{3}{2}}+\tfrac{189}{10}(1-x^2)^{\frac{5}{2}}-9(1-x^2)^{\frac{7}{2}} \quad \text{for} \quad N=4.\end{aligned} \tag{85}$$

Although the coefficients in these functions appear to be relatively simple rational numbers, no method has been found for computing them without explicitly evaluating N by N determinants.

According to (33) the assumption (60) should lead to the largest discrepancies from condition (11) at $t=0$. The deviations of the functions (85) from the value 1 are therefore a measure of the precision with which the assumption (60) and the conditions (76) and (77) approximate the original boundary conditions (11). The functions (85) are listed in table 1.

TABLE 1

	$N=1$	2	3	4
0	1·500	0·833	1·167	0·900
0·1	1·499	0·850	1·155	0·925
0·2	1·470	0·882	1·103	0·976
0·3	1·430	0·938	1·039	1·033
0·4	1·385	1·009	0·981	1·065
0·5	1·299	1·082	0·947	1·047
0·6	1·200	1·147	0·960	0·994
0·7	1·069	1·176	1·023	0·954
0·8	0·900	1·140	1·112	1·002
0·9	0·654	0·951	1·097	1·114
0·95	0·468	0·729	0·924	1·046
1·0	0·000	0·000	0·000	0·000

The case $N=0$ follows from (74), (78), and (82) and gives in accordance with (75) that $(1/w)\,\partial u/\partial t \sim \frac{1}{2}(1-x^2)^{-\frac{1}{2}}$ whose average is indeed equal to 1, but which is still far from the required constant value 1.

The deceleration which is experienced by an impinging rigid disk is large compared to g, if the kinetic energy of the disk corresponds to some reasonable height of fall and the semi-infinite elastic body does not have any exceptional values for the elastic constants. Although there is no particular difficulty connected with the weight term Mg in (77), this term will be neglected henceforth. The stress distribution (60) is then simply proportional to the initial speed w of the rigid disk. In terms of the variables x and y the only parameters of the theory are then the ratio k of (1) and the group of constants which occurs in the left-hand side of (78). If the rigid disk is replaced by a rigid cylinder of radius r_0, thickness h, and density ρ_d, and if ρ_s is the density of the semi-infinite elastic solid, then the quantity

$$q = \frac{2\rho_s r_0}{\rho_d h} \tag{86}$$

can be chosen as second parameter of the theory.

6. Resolution into modes

If the system of $N+1$ linear equations (78) and (79) is solved, the result can be written as

$$\frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} f_n(y) = -\frac{wy}{b} \frac{\tilde{G}_n(y)}{\sqrt{(2\pi)}\, G(y) + qH(y)}, \tag{87}$$

with the following notation

$$G_n(y) = (-1)^n \begin{vmatrix} H_{10}(y) & H_{11}(y) & \dots & H_{1n-1}(y) & H_{1n+1}(y) & \dots & H_{1N}(y) \\ H_{20}(y) & H_{21}(y) & \dots & H_{2n-1}(y) & H_{2n+1}(y) & \dots & H_{2N}(y) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{N0}(y) & H_{N1}(y) & \dots & H_{Nn-1}(y) & H_{Nn+1}(y) & \dots & H_{NN}(y) \end{vmatrix}, \tag{88}$$

$$G(y) = y^2 \sum_{n=0}^{N} G_n(y)\, \tilde{G}_n(y), \tag{89}$$

$$H(y) = \sum_{n=0}^{N} \frac{\tilde{G}_n(y)}{1.3\ldots(2n+1)}. \tag{90}$$

If the Laplace transform is inverted with the help of the ordinary complex inversion formula, the result is

$$\sigma = -\mu \frac{w}{b} \sum_{n=0}^{N} \frac{\sqrt{\pi}}{2^n\Gamma(n+\frac{1}{2})} \left(1-\frac{r^2}{r_0^2}\right)^{n-\frac{1}{2}} \frac{1}{2\pi \mathrm{i}} \int \frac{\tilde{G}_n(y)\, \mathrm{e}^{y(bt/r_0)}\, \mathrm{d}y}{\sqrt{(2\pi)}\, G(y) + qH(y)}, \tag{91}$$

$$\frac{\partial u}{\partial t} = w\pi \sum_{n=0}^{N} 2^{-n} \frac{1}{2\pi \mathrm{i}} \int \frac{\tilde{G}_n(y)\, \mathrm{e}^{y(bt/r_0)} y\, \mathrm{d}y}{\sqrt{(2\pi)}\, G(y) + qH(y)}\, h_n\left(\frac{r^2}{r_0^2}, y\right), \tag{92}$$

$$h_n(x^2, y) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda)}{\Gamma(n+1+\frac{1}{2}\lambda)} F(\tfrac{1}{2}-\tfrac{1}{2}\lambda, -n-\tfrac{1}{2}\lambda; 1; x^2) \frac{(-1)^\lambda P(\lambda)}{\Gamma(\lambda+1)} y^\lambda. \tag{93}$$

The path of the integration over y is parallel to the imaginary axis with a positive value for the real part of y.

The functions h_n, G_n, and $H_{\nu n}$ are holomorphic in the whole complex y plane. The only singularities in the integrands of (91) and (92) are poles which arise at the zeros of the denominator $\sqrt{(2\pi)}\, G(y) + qH(y)$. These zeros lie on the curves where

$$Q(y) = \sqrt{(2\pi)}\, G(y)/H(y) \tag{94}$$

becomes real and negative. The asymptotic expansions of G_n and $H_{\nu n}$ show that these functions have an oscillatory behaviour similar to the exponential function. There seem to exist a countable infinity of curves with a real negative value for $Q(y)$, and on each one of them $Q(y)$ takes on the particular value $-q$. Assume therefore that the equation

$$Q(y) = -q < 0, \quad \mathscr{I}(y) > 0 \tag{95}$$

has the solutions y_m with $m = 0, 1, \ldots$ in the upper half of the y plane, arranged according to increasing absolute value. A detailed examination of various asymptotic expansions (cf. the expansions quoted at the end of appendix D) shows that the integrals (92) and (93) can be evaluated by summing over the contributions from the poles y_m and their complex conjugates.

Therefore,

$$\sigma = -\mu \frac{w}{b} \sum_{m=0}^{\infty} \frac{e^{y(bt/r_0)}}{[H(y)]\,[dQ/dy]}\bigg|_{y_m} \left[\sum_{n=0}^{N} \frac{\sqrt{\pi}\tilde{G}_n(y_m)}{2^n\Gamma(n+\frac{1}{2})}\left(1-\frac{r^2}{r_0^2}\right)^{n-\frac{1}{2}}\right] + \text{compl. conj.} \tag{96}$$

$$\frac{\partial u}{\partial t} = w\pi \sum_{m=0}^{\infty} \frac{y\, e^{y(bt/r_0)}}{[H(y)]\,[dQ/dy]}\bigg|_{y_m} \left[\sum_{n=0}^{N} 2^{-n}\tilde{G}_n(y_m)\, h_n\left(\frac{r^2}{r_0^2}, y_m\right)\right] + \text{compl. conj.} \tag{97}$$

σ and $\partial u/\partial t$ are represented as sums over stress distributions and velocity distributions in brackets, each such distribution being multiplied by a time dependent factor of the type $\exp[y(bt/r_0)]$. The set of complex frequencies $r_0^{-1}by_m$ is obviously the same for σ and for $\partial u/\partial t$.

With the approximations implied in (60), (76), and (77) these complex frequencies solve the following problem: suppose that a rigid circular disk of radius r_0 and mass M, which is glued to the plane boundary of a semi-infinite elastic body, is driven by an external force in such a manner that the rigid disk performs a harmonic exponentially damped motion of the complex circular frequency $r_0^{-1}by$. For which values of y does the external driving force vanish for the case of perfect lubrication in the area of contact? The frequencies y_m and the corresponding stress distributions which would result from an exact treatment of the above problem are now called the modes of vibration for the rigid disk with perfect lubrication.

Even if all the modes were known from an exact treatment, only a finite number of them would be taken into account in a practical description of the motion of the impinging disk. Also, in a numerical computation N remains finite. It must therefore be investigated how many modes have to be obtained so that the motion of the impinging disk can be adequately described, and which modes can be adequately obtained with a given N.

A detailed examination of the asymptotic expansions for h_n, G_n, $H_{\nu n}$, and for the derived functions $\tilde{G}_n$, G, H would presumably give the answer. But rather than to go into these lengthy computations, which would be involved because N by N determinants have to be discussed, the author prefers to make the following conjectures. The real and imaginary parts of the complex frequency y_m increase at least linearly with respect to m. Therefore, only the lowest modes persist over a certain length of time. The stress distribution for a given mode y_m has m zeros so that for a given N only the modes $y_0, y_1 \ldots, y_N$ are adequately obtained, i.e. they lead to distributions of normal displacement with only small deviations from the average displacement. The other roots of equation (95) give only stress distributions and corresponding distributions of normal displacement whose sum leads to the initial distributions (75) and (85). The description of a certain mode is then not improved substantially by increasing N much beyond its order. But the initial amplitude and phase angle of that mode may still depend on N to a great extent, because they are mainly determined by the initial distributions of stress and normal displacement and these latter depend very much on N as was shown in the previous section.

7. Application of Cagniard's Method

The formula (62) for u^* is valid along the whole boundary of the elastic solid. A similar expression is found for v^* along the boundary. Formulae resembling (62) can also be obtained in a straightforward manner for u^* and v^* inside the elastic solid. Owing to the particular assumption (60) all these expressions can be evaluated by Cagniard's method (1939) even though the source of elastic waves is extended over a non-vanishing portion of the boundary. As an example the normal displacement along the boundary will be discussed.

Suppose first that the following decomposition has been performed.

$$\frac{\alpha}{(1+2\rho^2)^2-4\rho^2\alpha\beta} = \frac{(1+2\rho^2)^2\alpha+4\rho^2(k^{-2}+\rho^2)\beta}{1+8\rho^2+(24-16k^{-2})\rho^4+16(1-k^{-2})\rho^6}$$

$$= \frac{1}{\sqrt{(1+k^2\rho^2)}}\left(C_1-\frac{L_1}{1+l^2\rho^2}-\frac{M_1}{1+m^2\rho^2}-\frac{N_1}{1+n^2\rho^2}\right)$$

$$+\frac{1}{\sqrt{(1+\rho^2)}}\left(C_2-\frac{L_2}{1+l^2\rho^2}-\frac{M_2}{1+m^2\rho^2}-\frac{N_2}{1+n^2\rho^2}\right), \qquad (98)$$

for which the details can be found in appendix E. For the sake of mathematical rigour each term has to be considered separately. Each term $f(\rho)$ can be expanded in powers of ρ with coefficients γ_λ of which actually all the odd ones vanish.

$$f(\rho) = \sum_{\lambda=0}^{\infty} \gamma_\lambda \rho^\lambda, \quad \gamma_\lambda = 0, \quad \text{for} \quad \lambda = 1, 3, \ldots. \qquad (99)$$

The partial sums of the series (99) are called

$$S_\mu(\rho) = \sum_{\lambda=0}^{\mu-1} \gamma_\lambda \rho^\lambda. \qquad (100)$$

With the help of the formulae (cf. Whittaker & Watson 1952)

$$(\tfrac{1}{2}\zeta)^{-\nu-\frac{1}{2}} J_{\nu+\frac{1}{2}}(\zeta) = (-1)^{\nu+1}\frac{2^{2\nu+1}}{\sqrt{\pi}}\sum_{\mu=0}^{\nu}\frac{(\nu+\mu)!}{(\nu-\mu)!\,\mu!}\left[e^{i\zeta}(-2i\zeta)^{-\nu-\mu-1}+e^{-i\zeta}(2i\zeta)^{-\nu-\mu-1}\right], \qquad (101)$$

$$J_0(\zeta) = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{i\zeta\sin\omega}\,d\omega, \qquad (102)$$

the expression (62) can be written after some reduction for a typical term of (98) as

$$-r_0\sum_{\nu=0}^{N}(-4)^\nu\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}f_\nu(y)\sum_{\mu=0}^{\nu}\frac{(\nu+\mu)!}{(\nu-\mu)!\,\mu!}\frac{1}{2\pi i}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}d\omega$$

$$\times\int_0^\infty d\rho[f(\rho)-S_{\nu+\mu}(\rho)][e^{iy\rho(1+x\sin\omega)}(-2iy\rho)^{-\nu-\mu}-e^{-iy\rho(1+x\sin\omega)}(2iy\rho)^{-\nu-\mu}]$$

$$-r_0\sum_{\nu=0}^{N}(-4)^\nu\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}f_\nu(y)\sum_{\lambda=0}^{2\nu-1}\int_0^\infty\gamma_\lambda\rho^\lambda J_0(\rho xy)\,T_{\nu\lambda}(\rho y)\,d\rho, \qquad (103)$$

where the functions $T_{\nu\mu}(\zeta)$ are defined by

$$\left.\begin{aligned} T_{\nu\lambda}(\zeta) &= \frac{1}{2\mathrm{i}}\sum_{\mu=0}^{\nu}\frac{(\nu+\mu)!}{(\nu-\mu)!\,\mu!}[\mathrm{e}^{\mathrm{i}\zeta}(-2\mathrm{i}\zeta)^{-\nu-\mu}-\mathrm{e}^{-\mathrm{i}\zeta}(2\mathrm{i}\zeta)^{-\nu-\mu}] \quad \text{for} \quad \lambda<\nu, \\ T_{\nu\lambda}(\zeta) &= \frac{1}{2\mathrm{i}}\sum_{\mu=\lambda-\nu+1}^{\nu}\frac{(\nu+\mu)!}{(\nu-\mu)!\,\mu!}[\mathrm{e}^{\mathrm{i}\zeta}(-2\mathrm{i}\zeta)^{-\nu-\mu}-\mathrm{e}^{-\mathrm{i}\zeta}(2\mathrm{i}\zeta)^{-\nu-\mu}] \quad \text{for} \quad \lambda>\nu. \end{aligned}\right\} \tag{104}$$

All the integrals in (103) are convergent and the interchange of the ω integration with the ρ integration can be justified.

The integration over ρ in the second part of (103) can be performed with the help of (16) and the result is zero if $x>1$. The first part in (103) is now transformed according to Cagniard's method. First the ρ integration is deformed in the complex ρ plane so as to go along the positive or the negative imaginary axis in such a way that the argument of the various exponential functions always become real and negative. With the new variable $\rho'=\pm\mathrm{i}\rho$ and δ given by $x\sin\delta=1$, formula (103) becomes

$$-r_0\sum_{\nu=0}^{N}(-4)^{\nu}\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}f_\nu(y)\sum_{\mu=0}^{\nu}\frac{(\nu+\mu)!}{(\nu-\mu)!\,\mu!}\Big(\quad\Big),$$

$$\Big(\quad\Big)=\frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{-\delta}\mathrm{d}\omega\int_0^\infty\mathrm{d}\rho\,[f(\mathrm{i}\rho)-S_{\mu+\nu}(\mathrm{i}\rho)]\{-\exp[\rho xy(\sin\delta+\sin\omega)](-2y\rho)^{-\nu-\mu}-\dots\}$$
$$+\frac{1}{2\pi}\int_{-\delta}^{\frac{1}{2}\pi}\mathrm{d}\omega\int_0^\infty\mathrm{d}\rho[f(\mathrm{i}\rho)-S_{\mu+\nu}(\mathrm{i}\rho)]\{\exp[-\rho xy(\sin\delta+\sin\omega)](2y\rho)^{-\nu-\mu}+\dots\}. \tag{105}$$

The prime on ρ' has been omitted for simplicity, The second terms in the braces (indicated by a few dots) are formally identical to the first; but whereas the poles and branch points in the first terms are avoided by going above the real ρ axis, the poles and branch points in the second terms are avoided by going below the real ρ axis.

A new variable of integration is now introduced into the ρ integration, by putting $\theta=-x\rho(\sin\omega+\sin\delta)$ for $-(\frac{1}{2}\pi)\leqslant\omega<-\delta$ and by putting $\theta=x\rho(\sin\omega+\sin\delta)$ for $-\delta<\omega\leqslant(\frac{1}{2}\pi)$. After interchanging the θ integration with the ω integration and combining the two parts of the ω integration, expression (105) becomes

$$-r_0\sum_{\nu=0}^{N}(-1)^{\nu}\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}f_\nu(y)\sum_{\mu=0}^{\nu}\frac{(\nu+\mu)!\,2^{\nu-\mu}}{(\nu-\mu)!\,\mu!}y^{-\nu-\mu}\int_0^\infty\mathrm{e}^{-y\theta}\,\mathrm{d}\theta$$
$$\times\frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\mathrm{d}\omega\,[f(\mathrm{i}\rho)-S_{\mu+\nu}(\mathrm{i}\rho)]\,\rho^{-\nu-\mu}\frac{\partial\rho}{\partial\theta}\left\{\Big|_{\text{above }\rho\text{ axis}}\rho=\frac{\theta}{1+x\sin\omega}\Big|+\Big|_{\text{below }\rho\text{ axis}}\rho=\frac{\theta}{1+x\sin\omega}\Big|\right\}. \tag{106}$$

Instead of using the integration variable ω, it is again an advantage to use ρ with the understanding that θ remains constant as ρ varies from $\theta(1+x)^{-1}$ up to $+\infty$ and from $-\theta(x-1)^{-1}$ down to $-\infty$. The resulting integral can be written as a contour integral in the complex ρ plane if the integrand is made one-valued with the help of two cuts along the real axis: one from k^{-1}, 1, or $\theta(x+1)^{-1}$, whichever is smallest, to $+\infty$; and another one from $-k^{-1}$, -1, or $-\theta(x-1)^{-1}$, whichever is largest, to $-\infty$. The contour C starts at $\theta(x+1)^{-1}$, goes to $+\infty$ along the upper side of the real axis, swings around to $-\infty$ on a large circle in the upper half-plane, and comes finally up from $-\infty$ to $-\theta(x-1)^{-1}$ along the upper-side of the real axis. (106) becomes

$$-r_0 \sum_{\nu=0}^{N} \sum_{\mu=0}^{\nu} (-1)^\nu \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} f_\nu(y) \frac{(\nu+\mu)!\,2^{\nu-\mu}}{(\nu-\mu)!\,\mu!} y^{-\nu-\mu} \int_0^\infty e^{-y\theta}\,d\theta$$
$$\times \left\{ \frac{1}{2\pi i}(x^2-1)^{-\frac{1}{2}} \int_C [f(i\rho)-S_{\nu+\mu}(i\rho)] \frac{\rho^{-\nu-\mu}\,d\rho}{\sqrt{\{[\theta/(x+1)-\rho]\,[(\theta/(x-1)+\rho]\}}} + \text{compl. conj.} \right\} \tag{107}$$

where all the roots are defined to be positive for $\rho = 0$.

If all the terms of (98) are taken together, the integrand of the ρ integral has only the real poles at $\pm l^{-1}$ in the ρ plane with the cuts as prescribed above. The contour C can be deformed so as to run directly from $\theta(x+1)^{-1}$ down to $-\theta(x-1)^{-1}$ along the real axis. If therefore at $k\theta < x-1$ for the first terms or if $\theta < x-1$ for the second terms of the decomposition (98), the integral over ρ in (107) is real and the braces in (107) vanish. The θ integration in (107) goes only from Θ to ∞, where $k\Theta = x-1$ for the first terms and $\Theta = x-1$ for the second terms in (98). The contribution of $S_{\nu+\mu}$ $(i\rho)$ to (107) contains the expression

$$\sum_\mu \frac{(\nu+\mu)!\,{}^{\nu-\mu}}{(\nu-\mu)!\,\mu!} y^{-\nu-\mu} \int_\Theta^\infty e^{-y\theta}\,d\theta \left\{ \int_{\theta/(x+1)}^\infty - \int_{-\infty}^{-\theta/(x-1)} \right\} \gamma_\lambda \rho^\lambda \frac{\rho^{-\nu-\mu}\,d\rho}{\sqrt{\{[\rho-\theta/(x+1)]\,[\theta/(x-1)+\rho]\}}},$$

where the summation over μ goes from 0 to ν for $\lambda < \nu$, and from $\lambda-\nu+1$ to ν for $\lambda \geqslant \nu$. As $\lambda < 2\nu$ this can be evaluated directly and is then found to vanish. If therefore the lower limit of the θ integration is replaced by Θ, the term $S_{\nu+\mu}(i\rho)$ can be dropped altogether from the ρ integral in (107). The expansion (99) and the use of the polynomials (100) is now seen to be purely auxiliary, in order to insure the legitimacy of the various manipulations in this section; but it does not enter the final result. It is then possible to write

$$\frac{p}{b} u^* = -\sum_{\nu=0}^{N} \sum_{\mu=0}^{\nu} (-1)^\nu \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} f_\nu(y) \frac{(\nu+\mu)!\,2^{\nu-\mu}}{(\nu-\mu)!\,\mu!} y^{-\nu-\mu} W^*_{\nu+\mu}(y,x). \tag{108}$$

$W^*_\lambda(y,x)$ is the Laplace transform of the function $W_\lambda[(bt/r_0),(r/r_0)]$ which is defined as follows

$$W_\lambda(\theta,x) = W^{(1)}_\lambda(\theta,x) + W^{(2)}_\lambda(\theta,x), \tag{109}$$

with $W^{(1)}_\lambda(\theta,x)$ given by

$$\begin{aligned}
&= 0 \quad \text{for} \quad k\theta < x-1. \\
&= \frac{1}{\pi}\int_{\theta/(x+1)}^{k^{-1}} \frac{\rho^{-\lambda}\,d\rho}{\sqrt{(1-k^2\rho^2)}\sqrt{\{\rho^2x^2-(\theta-\rho)^2\}}} \left(C_1 + \frac{L_1}{l^2\rho^2-1} + \frac{M_1}{m^2\rho^2-1} + \frac{N_1}{n^2\rho^2-1}\right) \\
&\quad + \frac{L_1 l^{\lambda+1}}{2\sqrt{(k^2-l^2)}\sqrt{\{x^2-(l\theta-1)^2\}}} + \frac{l_1(-1)^{\lambda+1}}{2\sqrt{(k^2-l^2)}\sqrt{\{x^2-(l\theta-1)^2\}}} \quad \text{for} \quad x-1 < k\theta < x+1, \\
&= \frac{L_1 l^{\lambda+1}}{2\sqrt{(k^2-l^2)}\sqrt{\{x^2-(l\theta-1)^2\}}} + \frac{L_1(-l)^{\lambda+1}}{2\sqrt{(k^2-l^2)}\sqrt{\{x^2-(l\theta-1)^2\}}} \quad \text{for} \quad \frac{x+1}{k} < \theta < \frac{x+1}{l}, \\
&= 0 \quad \text{for} \quad l\theta > x+1,
\end{aligned} \tag{110}$$

and with $W^{(2)}_\lambda(\theta,x)$ given by a similar set of formulae which is obtained formally from (110) by putting $k=1$ and replacing the index 1 by 2. The integral in (110) is a complete elliptic integral whose evaluation is straightforward. The algebraic terms in (110) have to be omitted if the roots become imaginary.

The expression (108) is the Laplace transform of $(1/b)$ $(\partial u/\partial t)$ and appears as the product of two Laplace transforms whose original functions are known. With the help of (87) and (95) it follows that

$$\frac{1}{w}\frac{\partial u}{\partial t} = \sum_{\lambda=0}^{\infty}\sum_{\nu=0}^{N}\sum_{\mu=0}^{\nu}\frac{(-1)^{\nu}(\nu+\mu)!}{2^{\mu}(\nu-\mu)!\,\mu!}\left.\frac{\tilde{G}_{\nu}(y)\,y^{1-\nu-\mu}\,\mathrm{e}^{y(bt/r_0)}}{H(y)\,(\mathrm{d}Q/\mathrm{d}y)}\right|_{y_\lambda}\int_0^{bt/r_0}\mathrm{e}^{-y_\lambda\theta}\,W_{\nu+\mu}\left(\theta,\frac{r}{r_0}\right)\mathrm{d}\theta. \qquad (111)$$

The fact has been used that the power series expansion of $\tilde{G}_{\nu}(y)$ starts with $y^{2\nu}$. This follows directly from the definition of $\tilde{G}_{\nu}(y)$, as well as from (33). Formula (111) shows that $(1/w)$ $(\partial u/\partial t)$ becomes the sum of exponentially damped harmonics if $lbt > r+r_0$, i.e. as soon as sufficient time has elapsed to allow a Rayleigh wave to be propagated over the distance $r+r_0$.

8. Numerical evaluation

As the power series (72), (80), and (81) converge quite well, it is natural to use them for the numerical computation of u^*, G_n, and $H_{\nu n}$. The practical merit of the conditions (76) and (77) is that they lead to power series whose coefficients can be generated by three-term recursion formulas. It is tempting to use the asymptotic expansions in order to obtain expressions for u^*, G_n, and $H_{\nu n}$ which are useful in the numerical computation of the higher modes. The main obstacle against the use of the asymptotic expansions is that their coefficients cannot be obtained by simple recursion formulae.

The decomposition (98) can be written symbolically as

$$\frac{\alpha}{(1+2\rho^2)^2-4\rho^2\alpha\beta} = \Sigma\odot\frac{1}{(1+\gamma^2\rho^2)^{\epsilon-\eta+\frac{1}{2}}(1+\delta^2\rho^2)^{\eta}}, \qquad (112)$$

where $\odot$ stands for the coefficients C_1, $-L_1$, etc. and the numbers ϵ and η are determined for each term in (98) by the conditions

$$|\gamma| > |\delta|, \quad \left\{\begin{matrix}\epsilon = 1 \rightarrow \eta = \frac{1}{2} \quad \text{or} \quad \eta = 1, \\ \epsilon = 0 \rightarrow \eta = 0.\end{matrix}\right\} \qquad (113)$$

The details of assigning γ, δ, ϵ, η to the terms of (98) are given in appendix E. From the integral

$$\int_0^{\infty}\frac{\rho^s\,\mathrm{d}\rho}{(1+\gamma^2\rho^2)^{\epsilon-\eta+\frac{1}{2}}(1+\delta^2\rho^2)^{\eta}} = \frac{\Gamma(\epsilon-\frac{1}{2}s)\,\Gamma[\frac{1}{2}(s+1)]}{2\gamma^{s+1}\,\Gamma(\epsilon+\frac{1}{2})}F(\eta,\tfrac{1}{2}(s+1);\epsilon+\tfrac{1}{2};1-\delta^2/\gamma^2) \qquad (114)$$

and from (24) and (25) it follows that

$$P(s) = \Sigma\odot\frac{\Gamma(\epsilon-\frac{1}{2}s)}{2\gamma^{s+1}\Gamma(\epsilon+\frac{1}{2})\,\Gamma(-\frac{1}{2}s)}F(\eta,\tfrac{1}{2}(s+1);\epsilon+\tfrac{1}{2};1-\delta^2/\gamma^2). \qquad (115)$$

If s is a positive integer, say λ, the hypergeometric functions on the right-hand side can be generated by successive application of the well-known recursion formulae for the hypergeometric function starting from $\lambda = 0$, 1, 2, and 3. The recursion formulae are given in appendix F together with the hypergeometric functions for $\lambda = 0, 1, 2, 3$ which turn out to be elementary functions of the argument $[1-(\delta^2/\gamma^2)]$.

The hypergeometric functions $F(\frac{1}{2}-\frac{1}{2}\lambda,\ -n-\frac{1}{2}\lambda;\ 1,\ x^2)$ in (72) and (93) are polynomials in x^2, because one of the two first parameters $\frac{1}{2}-\frac{1}{2}\lambda$ or $-n-\frac{1}{2}\lambda$ is always a negative integer. For $\lambda = 0, 1, 2, 3$ these polynomials are easily obtained by writing down the hypergeometric series which defines $F(\frac{1}{2}-\frac{1}{2}\lambda,\ -n-\frac{1}{2}\lambda;\ 1;\ x^2)$. For higher values of the positive integer λ these polynomials can be generated by successive application of the three-term recursion formula which is derived in appendix F.

The digital computer at the author's disposal (Elecom 120) works normally with 8 decimals. From the asymptotic behaviour of the coefficients in the power series (80) and (81) it seemed unreasonable to expect reliable results for $|y| > 8$. Also, the asymptotic expansions for (80) and (81) seemed to indicate that not more than three solutions of (95) are within that distance from the origin. In view of the speculations at the end of §6 and because of the fact that the computing time is proportional to $(N+1)^2$, it was decided to put $N = 2$ in (60) and all the subsequent formulae. If one looks at (85) and its numerical values, the case $N = 2$ appears to be the first in which the boundary condition $(1/w)\ (\partial u/\partial t) = \text{const.}$ for $r < r_0$ is crudely approximated even at $t = 0$. No effort was made to study numerically the effect of N on the approximation of the first modes.

The computations were performed for $k^2 = 3, 4, 5, 6, 8, 10, 20, 100$. In all these cases not more than the first two modes could be found. The parameter q of (86) was varied in the range $0 \leqslant q \leqslant 4$. Equation (95) was solved indirectly, i.e. with a given value of k^2, and for fixed real (imaginary) part y' of y the imaginary (real) part y'' was determined by iteration such that the imaginary part of $Q(y)$ vanished. This gave for each value of k^2 and for each mode a set of complex frequencies y with corresponding values for q, between which a four-point interpolation could be made. All the numbers in the tables constitute therefore interpolated values.

In order to examine the precision with which the modes are described for $N = 2$, the normal displacements u^* given by (72) and (87) were computed as functions of r/r_0 for $k^2 = 3, 5, 8, 20$. The deviations of u^* from its mean value are always largest at $r = 0$. For the lowest (zero) mode and for $q \leqslant 4$ this deviation never exceeded 10^{-4} of the mean value. It appears therefore that the complex frequencies of the lowest modes are determined with at least that accuracy by taking $N = 2$. For the first mode the derivation at $r = 0$ is much larger and increases with decreasing value of q. For $q > 1$ the deviation does not exceed 10^{-1} of the average value, but it increases up to three times that ratio as q is around $\frac{1}{4}$. Nevertheless, it is conjectured that the complex frequencies of the first modes are actually determined with much better accuracy than it would appear from looking at the deviations of u^* from its average value for a given N. The reason for this confidence is, of course, that the determination of the complex frequencies for the various modes by equation (77) involves only the average normal displacement. The only way to settle this equation about the accuracy of the complex frequencies for the first mode would be to compute them for higher values of N. It must be realized, however, that the distribution of normal stress in the area of contact is known for each mode only with a relative accuracy equal to the relative deviation of the normal displacement from its average value.

The formulae (96), (97), and (111) are useful only if the summation over the modes does not have to be carried very far. In order to make the present calculations useful, one has to be able to show that only the lowest and first modes are important. This requires a good knowledge of the initial amplitude and phase angle for each mode. There is considerable doubt as to the accuracy of these quantities with the present low value of $N = 2$. But it is hoped that the present method determines the initial amplitudes and phase angles with a relative accuracy comparable to the relative deviation of the normal displacement from its average value. With this hypothesis it is possible to gain an idea about the convergence of the summation over all modes, such as in (96), (97), and (111). This convergence is, of course, slowest for $t = 0$. On the other hand, it may be expected that the normal velocity at time $t = 0$ is positive for each mode, and the sum of these normal velocities equals the velocity w of the impinging disk. If (97) is integrated over r with the help of appendix C and evaluated at $t = 0$, it follows with (89) and (94) that

$$\sum_{\lambda=0}^{\infty} S_\lambda = 1, \quad S_\lambda = 2\mathscr{R} \left. \frac{Q(y)}{y \mathrm{d}Q/\mathrm{d}y} \right|_{y=y} \tag{116}$$

where each term is positive. The value of the terms $\lambda = 0$ and $\lambda = 1$ (figure 1) together with their sum (figure 2) have been computed for $k^2 = 3, 5, 8, 20$ and $0 \leqslant q \leqslant 4$. It is felt from looking at the graphs of these quantities, that the convergence of the summation over the modes is indeed quite fair. This point of view is, of course, supported by the fact that during the numerical computations, the complex frequency of the third mode was found to lie outside a circle around the origin with radius 8, so that its contribution is probably very highly damped, compared to the first mode whose damping constant lies between 2 and 3.

It is not the intention of this report to record all the information which was obtained from the numerical calculations, but only what is directly related to the motion of the disk. None of the results concerning the stress distribution in the contact area will be listed and discussed, because they would be of interest mainly in calculating the waves generated by the impinging disk according to (111).

Table 2 gives the length of the half-period for the lowest mode taking as unit of time $2r_0/b$, i.e. the time it takes a shear wave in the solid to travel across the diameter of the disk. Table 2 gives the damping factor for the lowest mode, i.e. the fraction by which the amplitude is reduced in half a period of the damped oscillation. If we write the complex frequency $y = -y' + \mathrm{i}y''$, table 2 gives the value of $2\pi/y''$ and table 2 gives the value of $\exp[-\pi(y'/y'')]$. These results are given numerically rather than graphically, because the numbers for the same value of q but different values of k^2 are so close together that they could not be separated in a graph of reasonable size. This implies that at least for $k^2 \geqslant 3$ the complex frequency of the lowest mode depends essentially only on the parameter q of (86) and on the shear wave velocity of the elastic solid (through the time unit $2r_0/b$), but not on Poisson's ratio. This result seems plausible because the stress distribution for the same total force exerted on the disk does not depend on Poisson's ratio in the static limit as shown by (33).

If the average normal velocity is written as

$$w \sum_\lambda R_\lambda \exp(-\alpha_\lambda t) \cos(w_\lambda t - \phi_\lambda), \tag{117}$$

where the index λ refers to the various modes and w is the initial velocity of the impinging disk, then R_λ may be called the absolute value of the velocity amplitudes. The values of R_λ are plotted in figure 3 against q for various values of k^2. If R_0 and R_1 are comparable for large values of q, the first mode dampens out much faster. Indeed, the lowest mode is reduced during one-half of its period by a factor $\exp[-\pi(\alpha_0/\omega_0)]$, whereas the first mode is reduced in the same time interval by a factor $\exp[-\pi(\alpha_1/\omega_0)]$. The first mode is therefore reduced by an additional factor $\exp[-\pi(\alpha_1-\alpha_0)/\omega_0]$ compared to the lowest mode. The values of

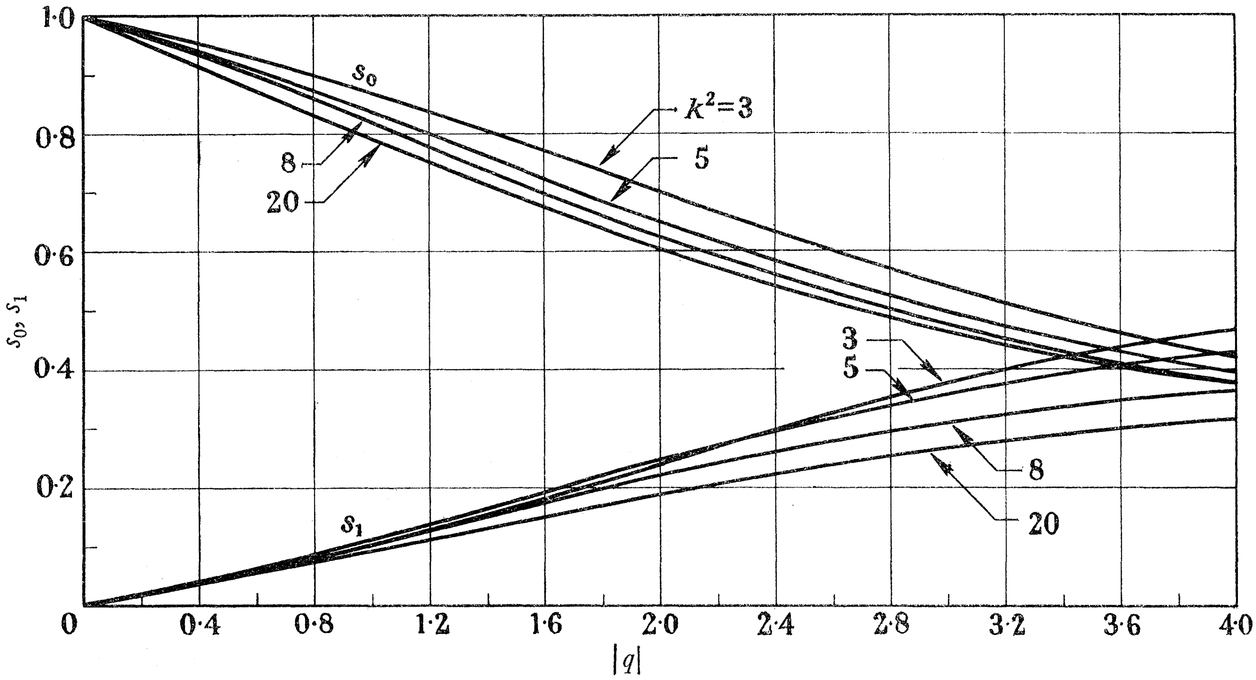

FIGURE 1. Velocity of the two lowest modes at time $t = 0$.

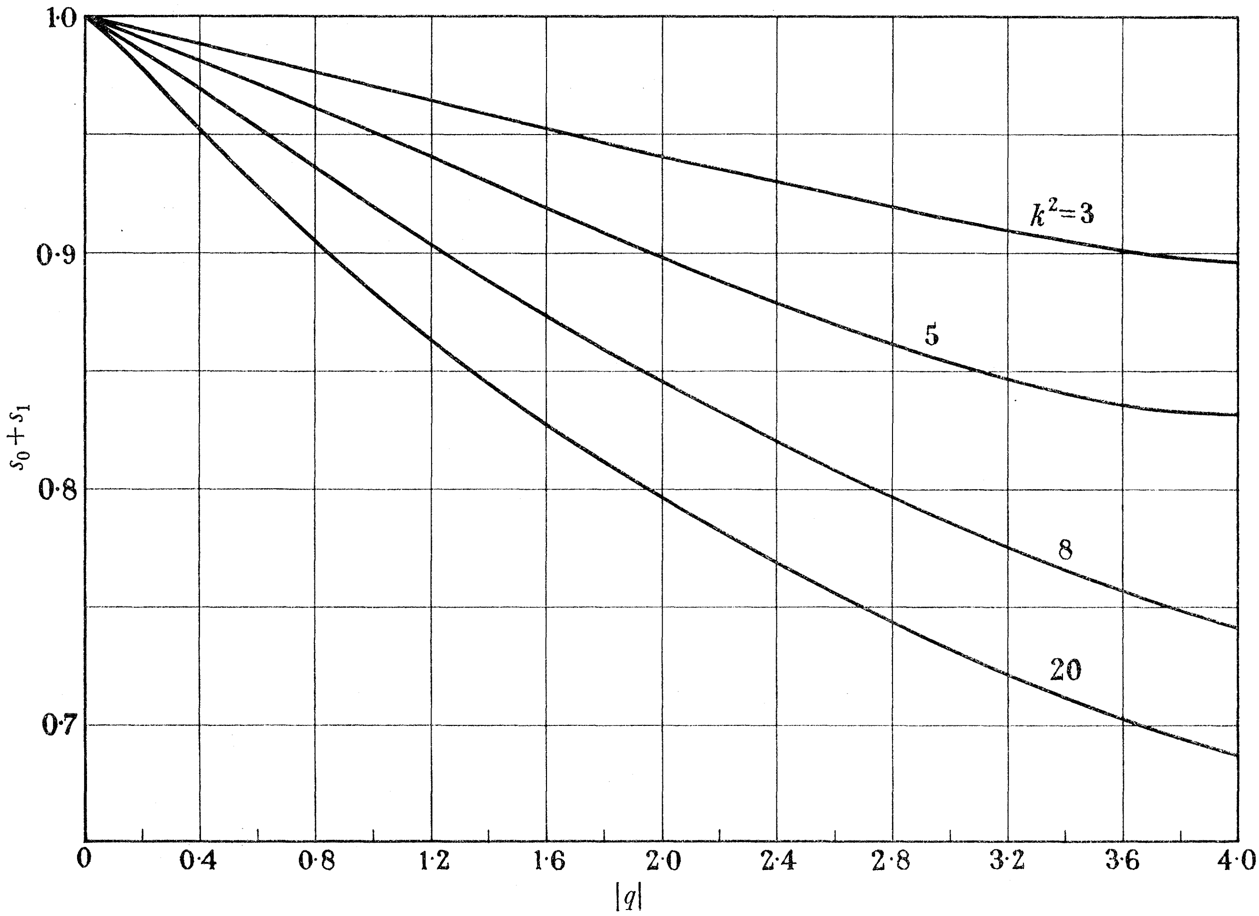

FIGURE 2. Total velocity at time $t = 0$ if only the two lowest modes are taken into account.

$\pi(\alpha_1-\alpha_0)/\omega_0$ are plotted in figure 4. It is evident that the contribution of the first mode to the average normal velocity has practically vanished after one half-period of the lowest mode.

The time derivative of (117) can be written as

$$-w\sum_{\lambda} R_\lambda \surd(\alpha_\lambda^2+\omega_\lambda^2)\exp(-\alpha_\lambda t)\sin(\omega_\lambda t+\phi_\lambda+\psi_\lambda), \quad \tan\psi_\lambda = \alpha_\lambda/\omega_\lambda. \tag{118}$$

Table 2

q	$k^2 = 3$	4	5	6	8	10	20	100
0·1	5·4557	5·1525	4·9951	4·8987	4·7872	4·7247	4·6095	4·5262
0·2	3·9058	3·6933	3·5845	3·5165	3·4443	3·4028	3·3313	3·2737
0·3	3·2287	3·0579	2·9714	2·9197	2·8615	2·8297	2·7674	2·7333
0·4	2·8318	2·6856	2·6130	2·5702	2·5220	2·4960	2·4503	2·4198
0·5	2·5654	2·4367	2·3734	2·3368	2·2959	2·2741	2·2365	2·2121
0·6	2·3725	2·2570	2·2008	2·1687	2·1334	2·1148	2·0833	2·0633
0·7	2·2257	2·1204	2·0702	2·0417	2·0106	1·9945	1·9677	1·9513
0·8	2·1098	2·0132	1·9678	1·9422	1·9148	1·9007	1·8775	1·8640
0·9	2·0162	1·9269	1·8853	1·8622	1·8380	1·8255	1·8056	1·7942
1·0	1·9391	1·8559	1·8179	1·7968	1·7749	1·7642	1·7470	1·7374
1·2	1·8201	1·7470	1·7146	1·6970	1·6793	1·6710	1·6578	1·6510
1·4	1·7334	1·6682	1·6406	1·6260	1·6115	1·6046	1·5944	1·5895
1·6	1·6685	1·6104	1·5861	1·5736	1·5615	1·5560	1·5481	1·5445
1·8	1·6192	1·5666	1·5453	1·5346	1·5245	1·5199	1·5137	1·5111
2·0	1·5812	1·5335	1·5146	1·5053	1·4967	1·4929	1·4879	1·4859
2·4	1·5292	1·4889	1·4738	1·4666	1·4602	1·4575	1·4540	1·4527
2·8	1·4982	1·4635	1·4510	1·4451	1·4401	1·4379	1·4352	1·4343
3·2	1·4803	1·4498	1·4390	1·4341	1·4299	1·4281	1·4259	1·4249
3·6	1·4709	1·4434	1·4339	1·4282	1·4258	1·4234	1·4223	1·4212
4·0	1·4670	1·4417	1·4330	1·4291	1·4257	1·4243	1·4223	1·4212

Table 3

q	$k^2 = 3$	4	5	6	8	10	20	100
0·1	0·6950	0·6823	0·6731	0·6665	0·6575	0·6517	0·6472	0·6287
0·2	0·5960	0·5804	0·5705	0·5612	0·5510	0·5443	0·5307	0·5182
0·3	0·5284	0·5115	0·4997	0·4912	0·4801	0·4731	0·4586	0·4465
0·4	0·4767	0·4580	0·4468	0·4381	0·4268	0·4198	0·4054	0·3936
0·5	0·4347	0·4165	0·4047	0·3955	0·3843	0·3773	0·3634	0·3521
0·6	0·3993	0·3809	0·3686	0·3601	0·3490	0·3423	0·3289	0·3182
0·7	0·3689	0·3505	0·3383	0·3300	0·3192	0·3128	0·2999	0·2899
0·8	0·3423	0·3241	0·3121	0·3039	0·2936	0·2876	0·2752	0·2658
0·9	0·3188	0·3007	0·2890	0·2811	0·2712	0·2652	0·2537	0·2449
1·0	0·2977	0·2799	0·2686	0·2606	0·2513	0·2457	0·2349	0·2267
1·2	0·2617	0·2446	0·2339	0·2268	0·2180	0·2130	0·2034	0·1964
1·4	0·2318	0·2154	0·2055	0·1990	0·1912	0·1866	0·1782	0·1721
1·6	0·2067	0·1913	0·1821	0·1761	0·1690	0·1650	0·1575	0·1523
1·8	0·1853	0·1709	0·1624	0·1569	0·1506	0·1470	0·1404	0·1359
2·0	0·1671	0·1535	0·1457	0·1408	0·1351	0·1318	0·1261	0·1221
2·4	0·1377	0·1259	0·1193	0·1152	0·1106	0·1080	0·1034	0·1004
2·8	0·1154	0·1052	0·0996	0·0959	0·0925	0·0904	0·0866	0·0844
3·2	0·0982	0·0894	0·0847	0·0819	0·0787	0·0770	0·0740	0·0721
3·6	0·0849	0·0770	0·0731	0·0707	0·0681	0·0666	0·0642	0·0626
4·0	0·0744	0·0675	0·0640	0·0621	0·0597	0·0585	0·0564	0·0551

It represents the average normal acceleration. Even for it the contribution of the first mode has practically vanished after a time interval which is of the order of one half-period of the lowest mode. The time at which the average normal acceleration vanishes is therefore essentially determined by the lowest mode alone. This time can be called the rebound time, because the disk would lose contact with the elastic solid as soon as the average normal acceleration changes sign, if the disk is not glued to the elastic solid. Relative to the half-

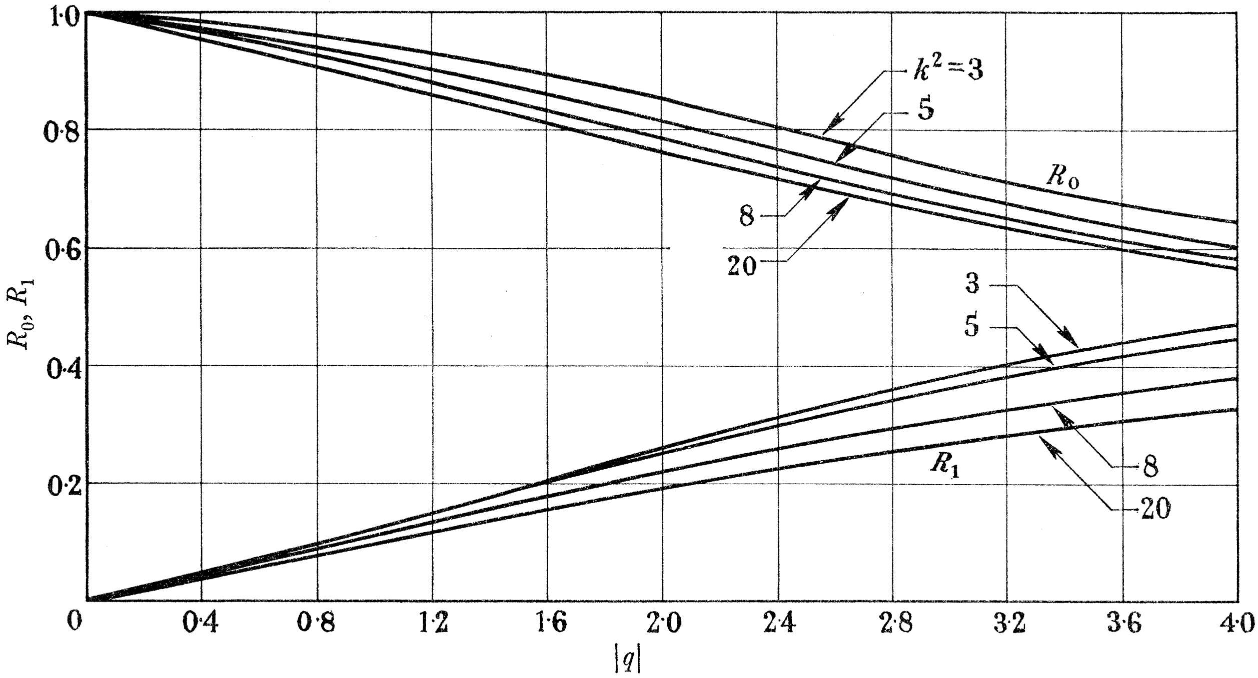

FIGURE 3. Velocity amplitude of the two lowest modes.

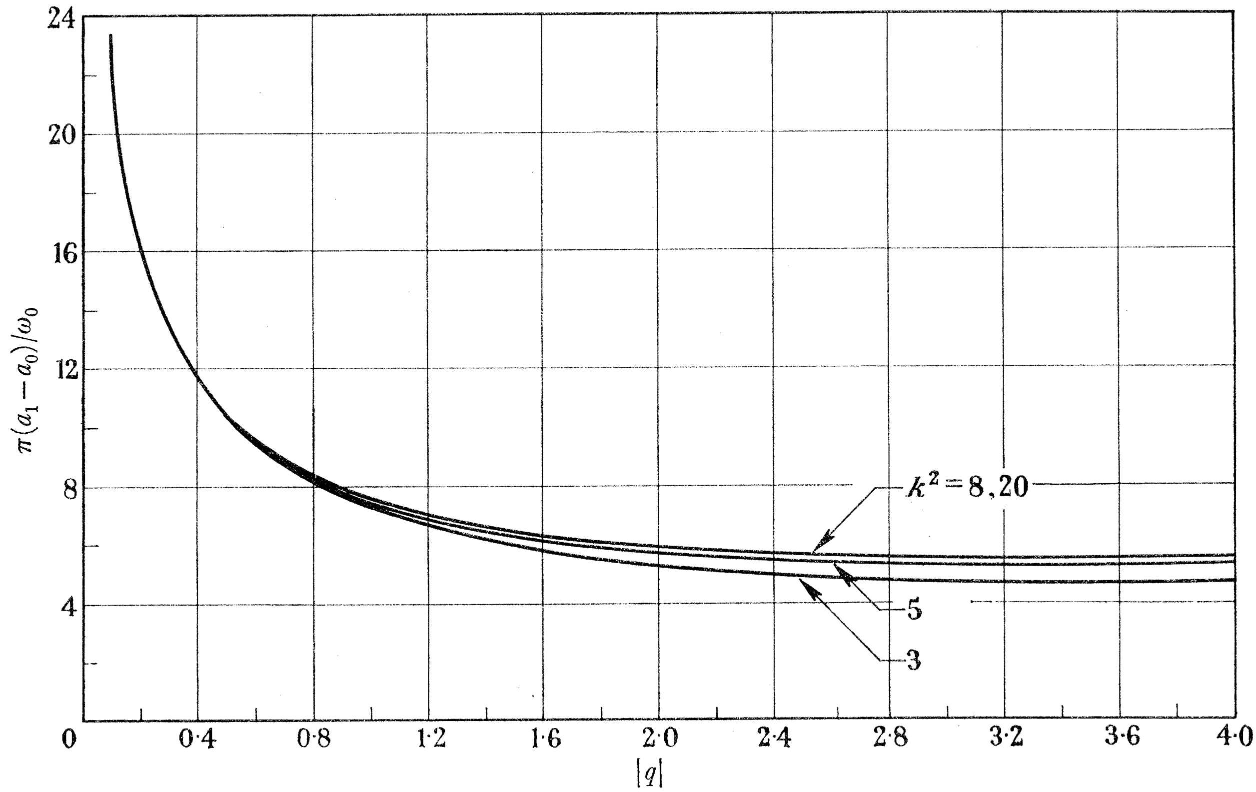

FIGURE 4. Natural logarithm of the ratio between the amplitudes of the lowest mode and the first mode after a half-period of the lowest mode.

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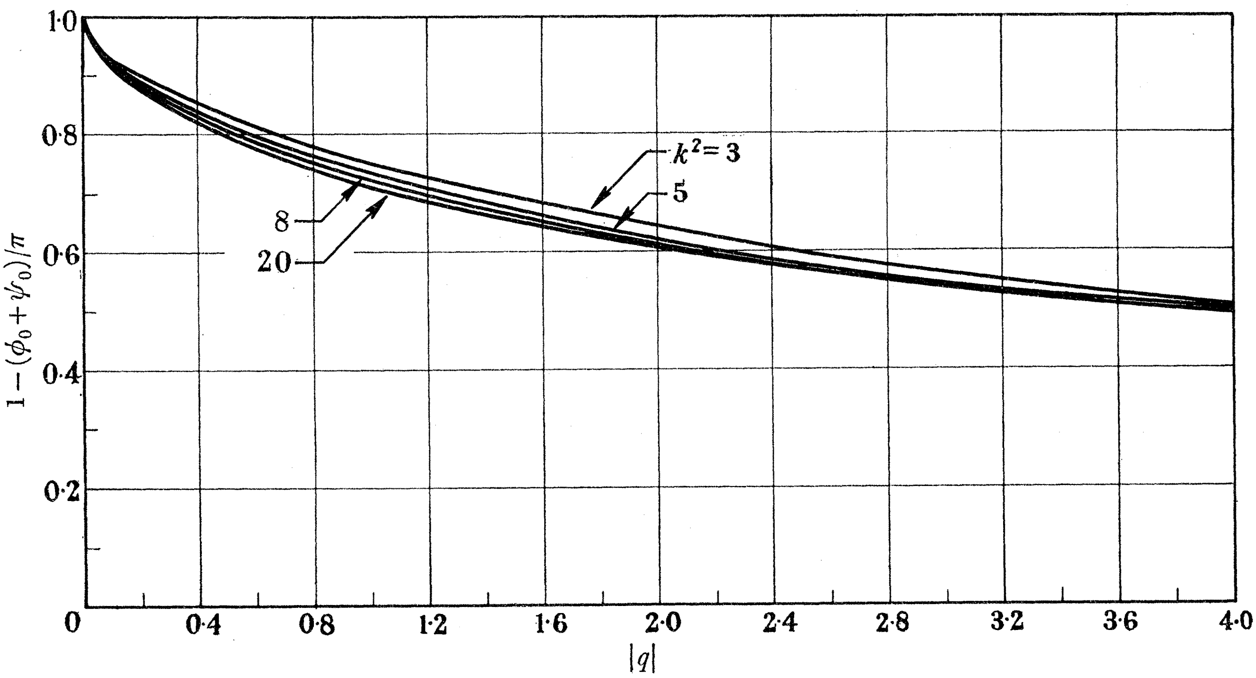

FIGURE 5. Rebound time of the disk relative to a half-period of the lowest mode.

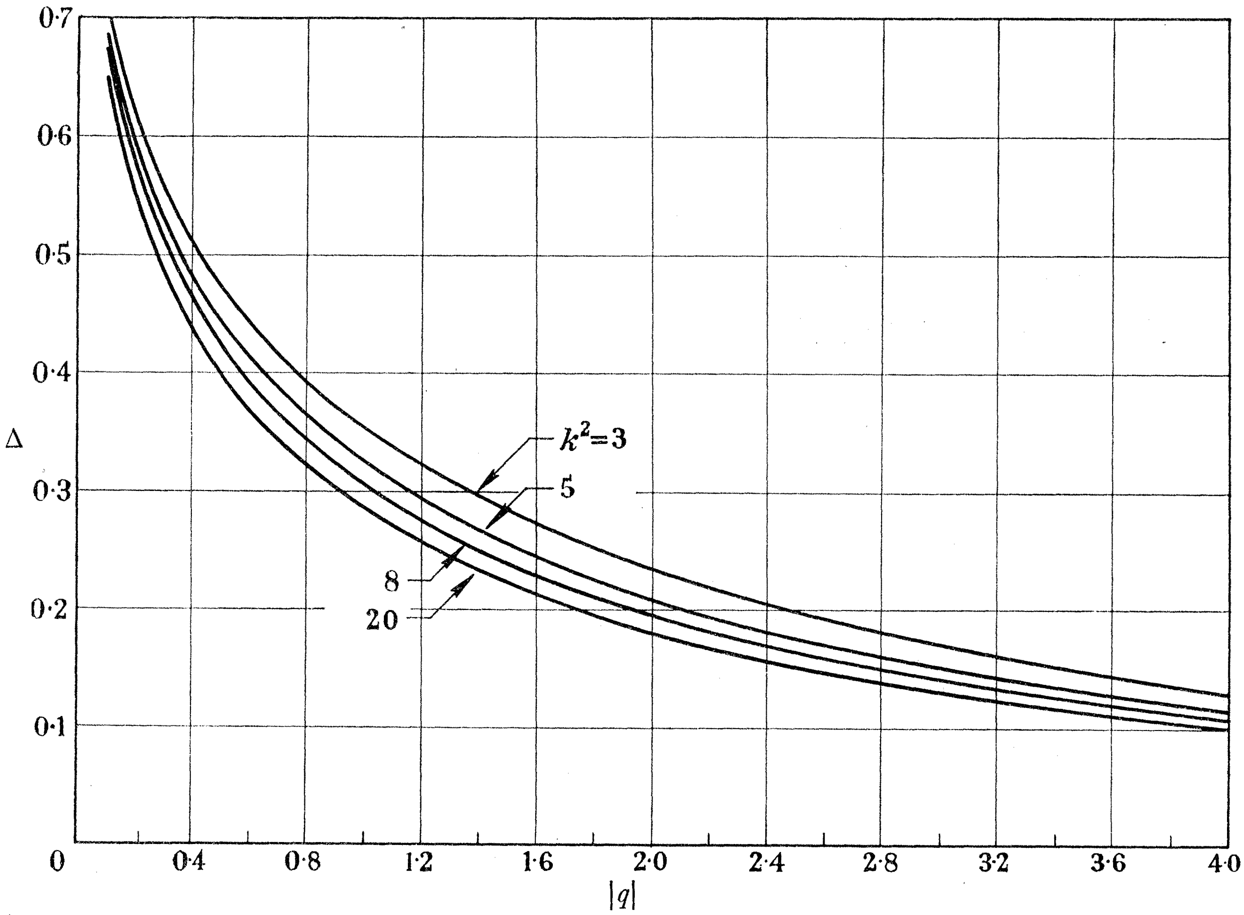

FIGURE 6. Ratio of rebound height to the height from which the disk was dropped.

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period of the lowest mode the rebound time is given by $1-[(\phi_0+\psi_0)/\pi]$, which is plotted in figure 5 against q for various values of k^2. The ratio Δ of the initial speed of the rigid disk to its speed at the time of rebound is given by

$$\Delta = \frac{R_0\alpha_0}{\sqrt{(\alpha_0^2+\omega_0^2)}} \exp\{-\pi(\alpha_0/\omega_0)\,[1-(\phi_0+\psi_0)/\pi]\}. \tag{119}$$

This is plotted in figure 6 against q for various values of k^2. If the rigid disk is assumed to have acquired its initial speed by dropping from a certain height, then it would rebound to a height whose ratio to the height of drop is given by the square of (119).

It is a pleasure to acknowledge the help of Miss E. E. Allen and of Dr A. S. Ginzbarg who wrote the program and performed all the calculations for this work on the Elecom 120 computer.

Appendix A

An upper bound for the matrix $(C'C)^m$ has to be found in order to investigate the convergence of the expansions (45). Therefore put $CC' = E-B$, where E is the unit matrix.

$$B_{lm} = \frac{1}{\pi^2(m-l)}\sum_{l+1}^{m}\frac{1}{\lambda+\frac{1}{2}} \quad \text{for} \quad m>l,$$

$$B_{ll} = \frac{1}{\pi^2}\sum_{l+1}^{\infty}\frac{1}{(\lambda+\frac{1}{2})^2} \quad \text{and} \quad B_{lm}=B_{ml}.$$

From the inequality

$$\sum_{l+1}^{m}\frac{1}{\lambda+\frac{1}{2}} < \int_{l+1}^{m+1}\frac{\mathrm{d}x}{x} = \ln(m+1)-\ln(l+1),$$

it follows that
$$B_{ml} < \frac{\ln(m+1)-\ln(l+1)}{\pi^2(m-l)}$$

which holds even for $m=l$ provided l approaches m continuously on the right-hand side. Define now two functions $B(\mu,\lambda)$ and $D(\mu,\lambda)$ by

$$B(\mu,\lambda) = B_{ml} \quad \text{for} \quad m\leqslant\mu<m+1, \quad l\leqslant\lambda<l+1;$$

$$D(\mu,\lambda) = \frac{\ln\mu-\ln\lambda}{\pi^2(\mu-\lambda)} \quad \text{for} \quad \mu\geqslant1, \quad \lambda\geqslant1;$$

$$D(\mu,\lambda) = D(\lambda,\mu) = \frac{\ln\mu}{\pi^2(\mu-1)} \quad \text{for} \quad \mu\geqslant1, \quad 0\leqslant\lambda\leqslant1;$$

$$D(\mu,\lambda) = \frac{1}{\pi^2} \quad \text{for} \quad 0\leqslant\mu\leqslant1, \quad 0\leqslant\lambda\leqslant1.$$

Because of the inequality $B(\mu,\lambda) < D(\mu,\lambda)$ and

$$(B^{n+1})_{mk} = \int_0^\infty \mathrm{d}\lambda_1 \ldots \int_0^\infty \mathrm{d}\lambda_n B(\mu,\lambda_1)\,B(\lambda_2,\lambda_1)\ldots B(\lambda_n,\kappa),$$

with $m\leqslant\mu<m+1$ and $k\leqslant\kappa<k+1$, it follows that

$$(B^{n+1})_{mk} < \int_0^\infty \mathrm{d}\lambda_1 \ldots \int_0^\infty \mathrm{d}\lambda_n D(m+1,\lambda_1)\,D(\lambda_2,\lambda_1)\ldots D(\lambda_n,k+1),$$

where the fact that the function $D(\mu, \lambda)$ is continuous has been used. The integrations over λ are split in two parts, from 0 to 1 and from 1 to ∞. In the first part $D(\mu, \lambda)$ is constant (equal to its value for $\lambda = 1$). This leads only the following integrals to be computed.

$$D^{(n)}_{\mu\kappa} = \frac{1}{\pi^{2n+2}} \int_1^\infty \mathrm{d}\lambda_1 \ldots \int_1^\infty \mathrm{d}\lambda_n \frac{\ln\lambda_1 - \ln\mu}{\lambda_1 - \mu} \frac{\ln\lambda_2 - \ln\lambda_1}{\lambda_2 - \lambda_1} \ldots \frac{\ln\kappa - \ln\lambda_n}{\kappa - \lambda_n}$$

with $\mu \geqslant 1$ and $\kappa \leqslant 1$. With the new variables

$$\rho = \ln\kappa, \quad \sigma = \ln\lambda, \quad \tau = \ln\mu$$

the integral $D^{(n)}_{\mu\kappa}$ becomes after some reduction

$$D^{(n)}_{\mu\kappa} = \left(\frac{1}{2\pi^2}\right)^{n+1} \mathrm{e}^{-\frac{1}{2}(\rho+\tau)} \int_0^\infty \mathrm{d}\sigma_1 \ldots \int_0^\infty \mathrm{d}\sigma_n \frac{\sigma_1 - \tau}{\sinh\frac{1}{2}(\sigma_1-\tau)} \frac{\sigma_2-\sigma_1}{\sinh\frac{1}{2}(\sigma_2-\sigma_1)} \cdots \frac{\rho-\sigma_n}{\sinh\frac{1}{2}(\rho-\sigma_n)}.$$

As it is sufficient to have an upper bound for $D^{(n)}_{\mu\kappa}$, insert the inequality

$$\frac{\rho-\sigma_n}{\sinh[\frac{1}{2}(\rho-\sigma_n)]} \leqslant 2$$

and the new variables of integration

$$\xi_1 = \sigma_1 - \tau, \quad \xi_2 = \sigma_2 - \sigma_1, \ldots, \quad \xi_n = \sigma_n - \sigma_{n-1},$$

whose limits are now extended from $-\infty$ to $+\infty$. Thus

$$D^{(n)}_{\mu\kappa} < 2\mathrm{e}^{-\frac{1}{2}(\rho+\tau)} \left(\frac{1}{2\pi^2}\right)^{n+1} \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \ldots \int_{-\infty}^{+\infty} \mathrm{d}\xi_n \frac{\xi_1}{\sinh\frac{1}{2}\xi_1} \cdots \frac{\xi_n}{\sinh\frac{1}{2}\xi_n},$$

where the integral

$$\int_{-\infty}^{+\infty} \frac{\xi\,\mathrm{d}\xi}{\sinh\frac{1}{2}\xi} = 8\int_0^\infty \frac{\eta\,\mathrm{e}^{-\eta}}{1-\mathrm{e}^{-2\eta}}\mathrm{d}\eta = 8\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right) = \pi^2$$

is used, so that finally for $\mu \geqslant 1$ and $\kappa \geqslant 1$

$$D^{(n)}_{\mu\kappa} < \frac{1}{2^n\pi^2\sqrt{\mu\kappa}}.$$

The following inequality is readily obtained

$$B(\mu,\lambda) = \frac{\ln\mu}{\pi^2(\mu-1)} < \frac{1}{\pi^2\sqrt{\mu}} \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1, \quad \mu \geqslant 1.$$

In combining these results it follows that

$$(B^{n+1})_{mk} < \frac{1}{\pi^2}\left(\frac{1}{2}+\frac{1}{\pi^2}\right)^n \frac{1}{\sqrt{(m+1)}\sqrt{(k+1)}} \quad \text{for} \quad m, k = 0, 1, 2, \ldots,$$

which is true even for $n = 0$, as follows immediately from the initial inequality of this appendix. From

$$(CC')^{N+1} = (E-B)^{N+1} < E + \binom{N+1}{1}B + \binom{N+1}{2}B^2 + \ldots$$

the inequality (46) follows at once.

Appendix B

Consider in the complex t plane the integral

$$\frac{1}{2\pi \mathrm{i}}\int_C (t-1)^{\alpha-\gamma}\,(t+1)^{-\beta+\gamma-1}\,(t-z)^{-\alpha}\,\mathrm{d}t$$

with arbitrary exponents α and β, and positive integer γ. The path surrounds the points 1 and z counterclockwise and is assumed to be initially far out on the real positive axis with the phase of $t-z$ between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ and the phases of $(t-1)$ and $(t+1)$ zero. If z is taken sufficiently close to 1, one can write

$$\frac{1}{2\pi \mathrm{i}}\int_C (t-1)^{-\gamma}\,(t+1)^{-\beta+\gamma-1}\left(1+\frac{1-z}{t-1}\right)^{-\alpha}\mathrm{d}t$$
$$= \sum_{m=0}^{\infty}\binom{-\alpha}{m}(1-z)^m\,\frac{1}{2\pi \mathrm{i}}\int_C (t-1)^{-\gamma-m}(t+1)^{-\beta+\gamma-1}\,\mathrm{d}t.$$

The integral on the right-hand side is $1/(\gamma+m-1)!$ times the $(\gamma+m-1)$th derivative of $(t+1)^{-\beta+\gamma-1}$ at $t=1$, so that the left-hand side becomes

$$\sum_{m=0}^{\infty}\binom{-\alpha}{m}(1-z)^m\binom{-\beta+\gamma-1}{\gamma+m-1}2^{-\beta-m} = \frac{2^{-\beta}\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)\,\Gamma(1-\beta)}\,F\left(\alpha,\beta;\gamma;\frac{1-z}{2}\right).$$

Now make the following specialization: Put $z=\cosh 2\xi$ with real $\xi>0$, $\alpha=n+\frac{3}{2}+\lambda$, with an integer $n\geqslant 0$, $\beta=-\lambda$, $\gamma=1$; choose as path of integration the circle with centre at $t=\cosh 2\zeta$ and radius equal to $\sinh 2\xi$. Introduce the variable θ by

$$t=\cosh 2\xi+\sinh 2\xi\,\mathrm{e}^{\mathrm{i}\theta},\quad (0\leqslant\theta\leqslant 2\pi).$$

The expression

$$\eta^2=\frac{t^2-1}{2(t-\cosh 2\xi)}=\cosh 2\xi+\sinh 2\xi\cos\theta$$

is then purely real and positive along the path of integration. This gives the integral representation

$$F(n+\tfrac{3}{2}+\lambda,\,-\lambda;1;\tfrac{1}{2}(1-\cosh 2\xi))=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{2\eta^2}{1+t}\right)^{n+\frac{1}{2}}\eta^{2\lambda}\,\mathrm{d}\theta.$$

The phase angle of

$$\frac{2\eta^2}{1+t}=\frac{1}{\cosh\xi}(\cosh\xi+\sinh\xi\,\mathrm{e}^{-\mathrm{i}\theta})$$

is always between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, and the values for θ and for $2\pi-\theta$ are complex conjugate. This determines the root of $2\eta^2(1+t)^{-1}$ uniquely as

$$2\surd(\cosh\xi)\,(\cosh\xi+\sinh\xi\,\mathrm{e}^{-\mathrm{i}\theta})^{\frac{1}{2}}=\surd\{(\eta+\mathrm{e}^{-\xi})\,(\mathrm{e}^{\xi}+\eta)\}\mp\mathrm{i}\surd\{(\eta-\mathrm{e}^{-\xi})\,(e^{\xi}-\eta)\}$$

with the upper sign for $0\leqslant\theta\leqslant\pi$, the lower for $\pi\leqslant 0\leqslant 2\pi$. The variable η is now used instead of θ as variable of integration, and the interval of integration is split into two parts at $\theta=\pi$. With

$$\mathrm{d}\theta=-\frac{2\eta\,\mathrm{d}\eta}{\sin 2\xi\sin\theta}=-\frac{2\eta\,\mathrm{d}\eta}{\surd\{(\mathrm{e}^{2\xi}-\eta^2)\,(\eta^2-e^{-2\xi})\}}$$

the integral representation (66) follows at once.

Appendix C

In computing the averages (76) and (77) it is advantageous to use the variable

$$\zeta = \surd(1-x^2), \quad \zeta \, d\zeta = -x \, dx.$$

If $a^{(m)} = a(a+1)\dots(a+m-1)$ and $a^{(0)} = 1$ with $a^{(m)} = 0$ as soon as one of the factors vanishes, one can write the following sequence of equations

$$\int_0^1 F(\tfrac{1}{2}-\tfrac{1}{2}\lambda, -n-\tfrac{1}{2}\lambda; 1; x^2)\, x\, dx = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}-\frac{1}{2}\lambda)^{(m)}(-n-\frac{1}{2}\lambda)^{(m)}}{1^{(m)}\,1^{(m)}} \int_0^1 x^{2m}\, x\, dx$$

$$= \frac{1}{2}\sum_{m=0}^{\infty} \frac{(\frac{1}{2}-\frac{1}{2}\lambda)^{(m)}(-n-\frac{1}{2}\lambda)^{(m)}}{1^{(m)}2^{(m)}} = \tfrac{1}{2}F(\tfrac{1}{2}-\tfrac{1}{2}\lambda, -n-\tfrac{1}{2}\lambda; 2; 1)$$

$$= \frac{\Gamma(n+2+\frac{3}{2})}{2\Gamma(\frac{1}{2}\lambda+\frac{3}{2})\,\Gamma(n+2+\frac{1}{2}\lambda)}.$$

In the last step the formula of Gauss for the hypergeometric function of unit argument has been used (cf. Whittaker & Watson 1952), which is also quoted in appendix F. The function (80) arises from inserting the last formula into (72) and using (25).

From the integral (cf. Erdelyi 1954)

$$\int_0^1 \zeta^{2l} P_{2\nu}(\zeta)\, d\xi = \frac{\Gamma(l+1)\,\Gamma(l+\frac{1}{2})}{2\Gamma(l-\nu+1)\,\Gamma(l+\nu+\frac{3}{2})}$$

which is assumed to vanish if $l-\nu+1$ is zero or a negative integer, it follows that

$$\int_0^1 (1-\zeta^2)^m P_{2\nu}(\zeta)\, d\zeta = \frac{1}{2}\sum_{l=\nu}^{m} \frac{\Gamma(m+1)\,\Gamma(l+\frac{1}{2})\,(-1)^l}{\Gamma(l-\nu+1)\,\Gamma(l+\nu+\frac{3}{2})\,\Gamma(m-l+1)},$$

which is again assumed to be zero for $m < \nu$. With the help of the elementary formula

$$\frac{1}{\Gamma(m-l+1)} = \frac{(-1)^l(-m)^{(l)}}{\Gamma(m+1)}$$

it follows again from Gauss's formula for the hypergeometric function of unit argument that

$$\int_0^1 (1-\zeta^2)^m P_{2\nu}(\zeta)\, d\zeta = \frac{(-1)^\nu\,\Gamma(m+1)\,\Gamma(\nu+\frac{1}{2})\,\Gamma(m+1)}{2\Gamma(m-\nu+1)\,\Gamma(\nu+1)\,\Gamma(\nu+m+\frac{3}{2})},$$

which again has to vanish for $m < \nu$. Therefore the equation

$$\int_0^1 F(\tfrac{1}{2}-\tfrac{1}{2}\lambda, -n-\tfrac{1}{2}\lambda; 1; 1-\zeta^2)\, P_{2\nu}(\zeta)\, d\zeta = \frac{(-1)^\nu\,\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)} \sum_{m=\nu}^{\infty} \frac{(\frac{1}{2}-\frac{1}{2}\lambda)^{(m)}(-n-\frac{1}{2}\lambda)^{(m)}}{\Gamma(m-\nu+1)\,\Gamma(\nu+m+\frac{3}{2})}$$

is obtained which is easily reduced to a hypergeometric function of unit argument. The preceding integral becomes

$$\frac{(-1)^\nu\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2})\,\Gamma(\frac{1}{2}\lambda+n+1)\,\Gamma(n+\lambda+1)}{2\Gamma(\nu+1)\,\Gamma(\frac{1}{2}\lambda-\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}\lambda+n-\nu+1)\,\Gamma(\nu+\frac{1}{2}\lambda+1)\,\Gamma(\nu+n+\frac{1}{2}\lambda+\frac{3}{2})}$$

and vanishes as soon as one of the arguments in the Γ functions of the denominator is zero or a negative integer. The functions (81) arise from inserting the last expression into (72) and using (25).

Appendix D

The method for deriving an asymptotic expansion is the same for $G_n(y)$ as for $H_{\nu n}(y)$. The computations will be performed explicitly only for the former. Equation (80) can be written as an integral of the Barnes type, namely

$$G_n(y) = \frac{1}{2\pi \mathrm{i}} \int \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{5}{4})\,\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(n+1+\frac{1}{2}s)\,\Gamma(n+2+\frac{1}{2}s)\,\Gamma(\frac{1}{2}s+\frac{3}{2})}\,\Gamma(-s)\,P(s)\,(2y)^s\,\mathrm{d}s,$$

where the path of integration in the complex s plane is restricted by the condition $-1 < \mathscr{R}(s) < 0$. $P(s)$ is now expressed by the integral (26) into which the decomposition (98) is inserted. The path of integration C in (26) can then be deformed at the same time as the cuts which join the branch points $\pm\mathrm{i}$ and $\pm\mathrm{i}k^{-1}$. The new cuts go from $+\mathrm{i}$ or $+\mathrm{i}k^{-1}$ along the positive imaginary axis to ∞ and from $-\mathrm{i}$ or $-\mathrm{i}k^{-1}$ along the negative imaginary to $-\infty$. The new path of integration comes from ∞ down along the positive imaginary axis on its negative side, goes around the branch point $+\mathrm{i}$ or $+\mathrm{i}k^{-1}$, and goes again up to ∞ along the positive imaginary axis on its positive side. A similar path surrounds the cut on the negative imaginary axis. After some obvious transformations to describe this new path, the expression for $P(s)$ becomes

$$P(s) = \frac{1}{\pi^2}\Gamma(1+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\cos\tfrac{1}{2}\pi\,(s+1)$$
$$\times\left[\int_{k^{-1}}^{\infty}\frac{\rho^s\,\mathrm{d}s}{\surd(k^2\rho^2-1)}\,(C_1-\dots)+\int_1^{\infty}\frac{\rho^s\,\mathrm{d}s}{\surd(\rho^2-1)}\,(C_2-\dots)\right].$$

The ρ integration can be interchanged with the s integration, leaving the following integral over s

$$\frac{1}{2\pi\mathrm{i}}\int\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{5}{4})\,\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(n+1+\frac{1}{2}s)\,\Gamma(n+2+\frac{1}{2}s)\,\Gamma(\frac{1}{2}s+\frac{3}{2})}$$
$$\times\Gamma(-s)\,\Gamma(1+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\sin\pi(1+\tfrac{1}{2}s)\frac{1}{\pi^2}(2y\rho)^s\,\mathrm{d}s.$$

With the help of the formulae (25) the integrand can be simplified somewhat and gives

$$\frac{1}{2\pi\mathrm{i}}\int\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n+\frac{1}{2}s+\frac{5}{4})}{\Gamma(n+1+\frac{1}{2}s)\,\Gamma(n+2+\frac{1}{2}s)}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)\,\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\pi^{\frac{3}{2}}(s+1)}\,(y\rho)^s\,\mathrm{d}s.$$

This integral converges as long as $|\arg y| < \frac{1}{2}\pi$, i.e. y has a positive real part (ρ is always real positive). It can be shown that under this condition an asymptotic expansion for this latter integral is obtained by simply pulling the path of s integration to the left across the poles of the integrand. The first of these poles is located at $s = -1$, so that the first term of an asymptotic expansion of $G_n(y)$ is given by the integral

$$\frac{\Gamma(\frac{1}{2}n+\frac{1}{4})\,\Gamma(\frac{1}{2}n+\frac{3}{4})}{\pi^{\frac{3}{2}}\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{3}{2})}\,\frac{1}{y}\left[\int_{k^{-1}}^{\infty}\frac{\rho^{-1}\,\mathrm{d}\rho}{\surd(k^2\rho^2-1)}\,(\mathrm{C}_1-\dots)+\int_1^{\infty}\frac{\rho^{-1}\,\mathrm{d}\rho}{\surd(\rho^2-1)}\,(C_2-\dots)\right],$$

whose value reduces indeed to (82) with the help of (25), if the poles at $\pm l$ are treated with sufficient care.

It is, of course, desirable to find asymptotic expansions for $G_n(y)$ and $H_{\nu n}(y)$ which are valid in the whole complex y plane. In order to do this the last s integral has to be rewritten in such a form as to be convergent in a wider domain of the variable $y\rho$. As this s integral represents essentially a generalized hypergeometric function of y (this follows if the path of the s integration is pulled to the right across the poles at $s = 1, 3, 5, \ldots$), it can be treated quite systematically by the method which was devised by Barnes (1906). This treatment is, however, quite clumsy in the present case, and must be applied with some caution, because logarithmic terms appear. It is sufficient to quote the result, i.e. the leading terms which occur in addition to (82), (83), and (84) for $\mathscr{R}y < 0$, namely,

$$G_n(y) \sim \pm \mathrm{i}(-1)^n\, l^{n+\frac{3}{2}} \sqrt{\pi}\, y^{-n-\frac{3}{2}} [L_1(k^2-l^2)^{-\frac{1}{2}} + L_2(1-l^2)^{-\frac{1}{2}}] \exp(-2y/l),$$

$$H_{\nu n}(y) \sim (-1)^{n+1}\, l^{n+1} \sqrt{\pi}\, y^{-n-1} [L_1(k^2-l^2)^{-\frac{1}{2}} + L_2(1-l^2)^{-\frac{1}{2}}] \exp(-2y/l).$$

The double sign is connected with the condition $|\arg y \pm \frac{1}{2}\pi| < \pi$.

Appendix E

The denominator in the second expression of (98) is written as

$$\begin{aligned} 1+8\rho^2+(24-16k^{-2})\,\rho^4+16(1-k^{-2})\,\rho^6 \\ = 16(1-k^{-2})\,(\rho^2+l^{-2})\,(\rho^2+m^{-2})\,(\rho^2+n^{-2}) \\ = (1+l^2\rho^2)\,(1+m^2\rho^2)\,(1+n^2\rho^2) \end{aligned}$$

with the convention $l^2 < |m^2| \leqslant |n^2|$. The coefficients of the decomposition (98) are then given explicitly by

$$C_1 = \frac{k}{4(1-k^{-2})}, \quad C_2 = \frac{1}{4(1-k^{-2})},$$

$$X_1 = \frac{k^{-3}}{4(1-k^{-2})}\,\frac{(k^2-2)^2x^4-(k^4-3k^2+4)\,x^2+8(k^4-2k^2+2)}{(x^2-y^2)\,(x^2-z^2)},$$

$$X_2 = \frac{k^{-2}}{4(1-k^{-2})}\,\frac{k^4x^4-8(k^4-2k^2+2)\,x^2+8(k^4-2k^2+2)}{(x^2-y^2)\,(x^2-z^2)},$$

where (x, y, z) is a cyclic permutation of (l, m, n).

There are two essentially different cases, which both occur in practice.

$$(1) \quad l^2 < 1 < k^2 < |m^2| < |n^2|,$$

$$(2) \quad l^2 < 1 < |m^2| \leqslant |n^2| < k^2.$$

The distribution of γ, δ, ϵ, η is the following (each term being characterized by its coefficient):

	first case				second case			
C_1	0	0	k^2	—	0	0	k^2	—
C_2	0	0	1	—	0	0	1	—
L_1	1	1	k^2	l^2	1	1	k^2	l^2
L_2	1	1	1	l^2	1	1	1	l^2
M_1	1	$\frac{1}{2}$	m^2	k^2	1	1	k^2	m^2
M_2	1	$\frac{1}{2}$	m^2	1	1	$\frac{1}{2}$	m^2	1
N_2	1	$\frac{1}{2}$	n^2	k^2	1	1	k^2	n^2
N_2	1	$\frac{1}{2}$	n^2	1	1	$\frac{1}{2}$	m^2	1

Appendix F

The general recursion formula

$$(c-b)\,F(a,b-1;c;x)+(2b-c-bx+ax)\,F(a,b;c;x)+b(x-1)\,F(a,b+1;c;x)=0$$

can be proved directly by inserting the hypergeometric series for each one of the hypergeometric functions. The interesting initial values are the following:

$$F(1,1;\tfrac{3}{2};x)=\frac{\arc\sin\sqrt{x}}{\sqrt{x}\sqrt{(1-x)}},\qquad F(1,2;\tfrac{3}{2};x)=\frac{1}{2(1-x)}\left(\frac{\arc\sin\sqrt{x}}{\sqrt{x}\sqrt{(1-x)}}-1\right),$$

$$F(1,\tfrac{3}{2};\tfrac{3}{2};x)=\frac{1}{1-x},\qquad F(1,\tfrac{5}{2};\tfrac{3}{2};x)=\frac{1-(\tfrac{1}{3}x)}{(1-x)^2},$$

$$F(\tfrac{1}{2},1;\tfrac{3}{2};x)=\frac{1}{2\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}},\qquad F(\tfrac{1}{2},2;\tfrac{3}{2};x)=\frac{1}{4\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}+\frac{1}{2(1-x)},$$

$$F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{3}{2};x)=\frac{1}{\sqrt{(1-x)}},\qquad F(\tfrac{1}{2},\tfrac{5}{2};\tfrac{3}{2};x)=\frac{1-(\tfrac{2}{3}x)}{(1-x)^{\frac{3}{2}}}.$$

In order to compute $F(\frac{1}{2}-\frac{1}{2}\lambda,\,-n-\frac{1}{2}\lambda;1;x^2)$ for large values of λ, a recursion formula will now be established which connects the three hypergeometric functions

$$F(a-1,b-1;c;x),\ F(a,b;c;x),\ \text{and}\ F(a+1,b+1;c;x).$$

A formula of the following type is assumed

$$(1-x)^2F(a+1,b+1;c;x)+(A+Bx)\,F(a,b;c;x)+CF(a-1,b-1;c;x)=0.$$

If the hypergeometric functions are replaced by the corresponding hypergeometric series, the coefficients of each power of x must vanish identically. Taking in particular x^{n+1}, the following condition arises

$$\begin{aligned}&(a+n)\,(a+n+1)\,(b+n)\,(b+n+1)-2(a+n)\,(b+n)\,(c+n)\,(n+1)\\&\quad+(c+n+1)\,(c+n)\,n(n+1)+Aab\,(a+n)\,(b+n)+Bab(c+n)\,(n+1)\\&\quad+C(a-1)\,a(b-1)\,b=0,\end{aligned}$$

which has to hold for all integers $n\geqslant -1$. As the terms in n^3 and n^4 cancel each other, the numbers A, B, C have to be determined as functions of a, b, c in such a manner that the terms in n^2, n^1, and n^0 vanish. This leads to three linear equations in A, B, C with the determinant given by

$$\begin{vmatrix}a^2b^2 & abc & ab(a-1)\,(b-1)\\ ab(a+b) & ab(c+1) & 0\\ ab & ab & 0\end{vmatrix}=a^3b^3(a-1)\,(b-1)\,(a+b-c-1).$$

As in the present case $c=1$ and $a<0$, $b<0$, this determinant is different from zero and the numbers A, B, C can be determined.

A simpler system of equations can be obtained by considering the following special cases of the assumed formula.

(1) Consider the terms in x^0.

(2) Put $x = 0$ using $F(a, b; c; 0) = 1$.

(3) Put $x = 1$ using the formula of Gauss.

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

(cf. Whittaker & Watson 1952). This gives the three following equations

$$Aa^2b^2+Babc+Cab(a-1)\,(b-1) = -ab(a+1)\,(b+1)+2abc,$$
$$A+C = -1,$$
$$A(c-a)\,(c-b)+B(c-a)\,(c-b)+C(c-a-b)\,(c+1-a-b) = 0,$$

whose determinant is equal to $a^2b^2(1+c-b-a)$. The resulting recursion formula is given by

$$F(a-1, b-1; c; x) = -\frac{c-b-a+1}{c-b-a-1}\,\frac{a}{c-a}\,\frac{b}{c-b}\,(1-x)^2F(a+1, b+1; c; x) + \left[\left(1+\frac{c-b-a+1}{c-b-a-1}\,\frac{a}{c-a}\,\frac{b}{c-b}\right)(1-x) + \frac{(c-a-b)\,(c-a-b+1)}{(c-a)\,(c-b)}\,x\right]F(a, b; c; x).$$

In the hypergeometric function of interest, $a = \frac{1}{2}-\frac{1}{2}\lambda$, $b = -n-\frac{1}{2}\lambda$, and $c = 1$.

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